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THE
THEORY OF STRUCTURES

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PREFACE TO SECOND EDITION

The First Edition of this book has been used by the author for the last four years as a text-book in large classes containing many mature students. Such errors and obscurities as have been discovered during that period have been eliminated in this edition. Additional matter, including an extension of the treatment of swing bridges, and chapters upon the Theorem of Least Work, Earth Pressure, Masonry Dams and Masonry Arches have been inserted. A few new problems for solution have also been added.

The author wishes to express his appreciation of the valuable assistance and suggestions he has received from some of the past and present members of the Civil Engineering Instructing Staff at Technology, and from certain graduate students. He is particularly indebted to Mr. Howard B. Luther, S. B., Dipl.-Ing., instructor in Civil Engineering.

August 15, 1915.

CHARLES M. SPOFFORD.

PREFACE TO FIRST EDITION

THE purpose of this book is to present in a thorough and logical manner the fundamental theories upon which the design of engineering structures is based and to illustrate their application by numerous examples. No attempt has been made to treat of the design of complete structures, but the design of the more important elements of which all structures are composed is fully considered.

The subject-matter is confined almost entirely to the treatment of statically determined structures, it being the writer's purpose to deal with indeterminate cases in another volume; the commonly used approximate methods for some of the more ordinary types of indeterminate structures are, however, included.

While the theories presented are for the greater part only such as have been in common use for many years, the method of treatment frequently differs considerably from that found in other books. Special attention may be called to the early introduction of the influence line and to its use in deriving and illustrating analytical methods, as well as to the chapter upon deflections.

The author wishes particularly to acknowledge his indebtedness to Professor George F. Swain for the logical and inspiring instruction received from him as a student.

July 18, 1911.

CHARLES M. SPOFFORD.

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EARTH PRESSURE

STRUCTURES

CHAPTER I

OUTER AND INNER FORCES

1. Definitions. A structure as defined in the "Century Dictionary" is, "a production or piece of work artificially built up, or composed of parts joined together in some definite manner." As used in this book, however, its meaning will be restricted to a part or combination of parts constructed to hold *in equilibrium* definite forces, with special reference to bridges and buildings.

Structures may be either *statically determined* or *statically undetermined.* Statically determined structures are those in which the reactions and primary stresses can be computed by statics. Structures for which these functions cannot be obtained by statics belong to the second class.

A bridge is a structure built to provide transportation across some natural or artificial obstacle, such as a river, ravine, street or railway. The term includes not only the superstructure of wood, metal or masonry, but also the substructure which may consist of masonry piers and abutments, or of steel towers. The superstructure may consist of simple beams supported at the ends directly on the masonry, or in case of long spans supported on cross beams which are themselves supported at the ends by girders, trusses or arches.[1] In the latter case the longitudinal beams are known as *stringers* and the cross beams as *floor beams.* As a clear conception of the function of the stringers

[1] For a clear understanding of girders and trusses see Figs. 1 to 4 and Arts. 60 and 78.

and floor beams is essential to the understanding of the matter which follows, the student is advised to study carefully Figs. 1, 2, 3 and 4, and to examine some of the bridges in his vicinity.

Deck bridges are those in which the floor is at the top of the

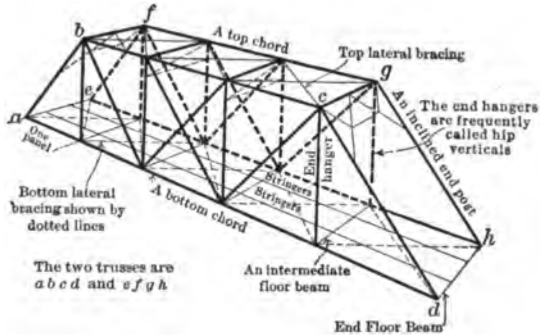


FIG. 1.—Framework of a Through Railroad Truss Bridge.

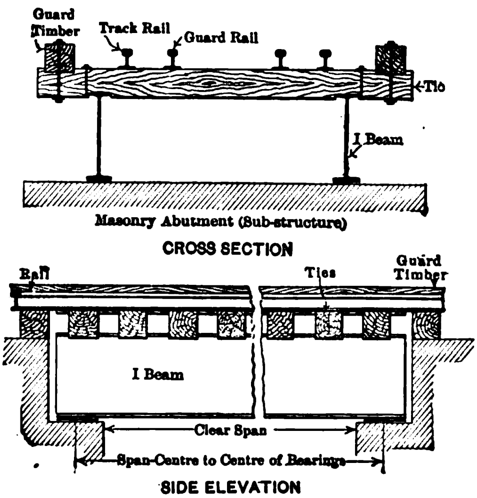


FIG. 2.—I-Beam Bridge for Single Track Railroad. (A Deck Structure.)

main superstructure, as in the simple I-beam bridge shown in Fig. 2. Such bridges if of considerable width require the use of floor beams and stringers, which may often be omitted for narrow structures.

Half-through bridges are those having the greater part of the superstructure above the floor level but with insufficient

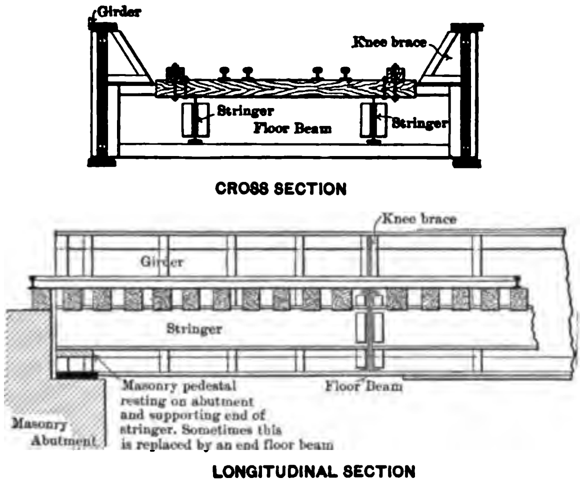


FIG. 3.—Half-Through Single Track Plate Girder Railroad Bridge.

Note.—Portion of bridge between floor beams measured along axis of bridge is called a panel. Stringers are considered as end-supported beams one panel in length.

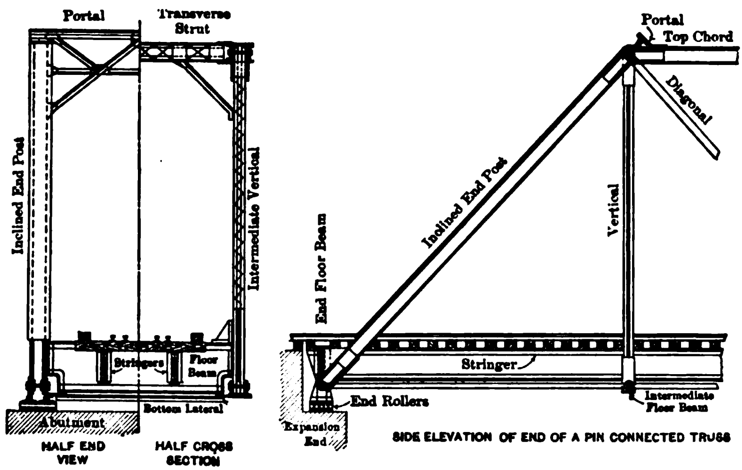


FIG. 4.—A Single Track Through Railroad Bridge.

depth to permit the use of overhead bracing. Lateral stability in such bridges may be obtained by the use of brackets or knee braces, as shown in Fig. 3.

Through bridges are those in which the greater part of the superstructure is above the floor level and in which overhead lateral bracing may be used between the trusses to obtain lateral stability. Such a bridge is shown by Fig. 4.

Whether a deck or through bridge should be used for a given location depends upon the external conditions. In general, bridges of considerable span are built as through structures unless the approaches on either side are at a considerable elevation above the obstacle to be crossed. The solution of this question for a given case is usually obvious and will not be considered here.

2. Live and Dead Loads. The forces to be considered may be divided into two classes: *outer* and *inner*. The outer forces consist of the applied loads and the resultant reactions, and may be divided into two distinct types: live or moving loads and dead or quiescent loads. The inner forces are the molecular forces which are brought into action by the outer forces and hold them in equilibrium. The dead load includes the weight of the structure itself, and all of its permanent quiescent load such as the pavement on highway bridges; the rails and other track appurtenances on railroad bridges; the floors, walls, roofs and partitions in buildings. The live load consists of all forces which are applied intermittently. For bridges these may be locomotives and cars, vehicles, pedestrians, snow and wind; for buildings they consist of people, snow, wind, office furnishings and partitions; for dams and retaining walls of water pressure and earth pressure.

3. Outer Forces. The determination of the intensity, distribution, and point of application of the outer forces is often difficult and requires mature judgment based upon extensive experience. For structures of great magnitude the question is particularly complicated and of vital importance; the design of such structures should never be attempted without a thorough study of this problem in its relation to the structure in question. In the following articles some of the difficulties in the way of an exact solution of the question will be presented and data given for use in the solution of the more common cases.

4. Weight of Structure. It is impossible to determine accurately the weight of a given structure before the completion of the design. It is equally impossible to design the structure with precision until its weight is known. It is therefore neces-

sary in all cases to make use of approximate methods of solution, first assuming the weight, next designing with the assumed data, then computing the weight and revising the design in the light of the new information thus obtained. For the more common types of structure data accumulated by experience may be used by the designer, and the first assumption made with sufficient accuracy to make revision unnecessary. For structures out of the ordinary, and particularly those in which the weight of the structure itself is a large percentage of the total load, several revisions are sometimes necessary, and a final computation of the weight after the completion of the detailed drawings and before the commencement of shop work, should never be omitted. The failure to do this for the huge Quebec bridge which failed during erection in 1907, resulted in serious errors in the stresses for which the structure was designed.

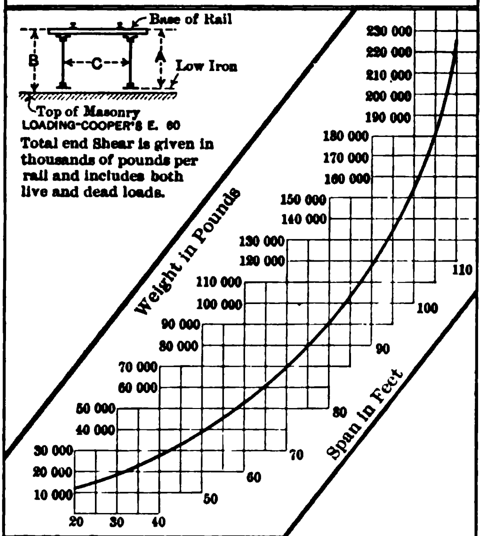
In all cases the designer should first design completely the minor portions of the structure and determine their weight carefully so as to eliminate as much uncertainty as possible. For example, in the design of a railroad bridge, the stringers should first be figured and their weight carefully determined, the floor beams may then be designed, and finally the lateral bracing, thus giving considerable information as to the total weight of the bridge and throwing the uncertainty into the main girders or trusses.

5. Weight of Railroad Bridges. It is possible to make a more accurate preliminary estimate of the weight of such bridges than can be done for other types of important structures, since there is less variation in loads and other conditions. Current practice on first class American railroads differs but little, and it is believed that the diagrams given in Figs. 5 to 10 inclusive give reasonable values for the *total weight of the steel in such structures.* The total weight of the bridge includes also the weight of the ties, rails and other accessories, which should be added to the values given in the diagrams. For the ordinary railroad bridge-floor with wooden ties this weight may be taken, in the lack of a specific design, as from 400 to 450 lbs. per linear foot. For solid ballasted floors this weight is of course much greater.

These diagrams were furnished by the Heath & Milligan Mfg. Co., Paint and Color Makers, of Chicago, Ill., U. S. A., for whom they were prepared by consulting engineers con-

nected with one of the large railroad systems of the country, and are for carbon-steel bridges designed for the typical locomotives shown in Fig. 11, and are known as Cooper's E_{60} loading. Where other loadings are to be used, these weights

DECK PLATE GIRDER SPANS
WEIGHTS, DIMENSIONS, ETC.



Span	Total End S.	A	B	C
30 ft.	102.2	4′11″	4′11″	7′0″
35 "	112.6	5-3	5-3	"
40 "	124.2	5-10	5-10	"
45 "	135.6	6-4	6-4	"
50 "	145.7	6-10	6-10	"
55 "	155.1	7-1	7-1	"
60 "	165.2	7-4	7-11	"
65 "	181.2	8-6	9-1	8′0″
70 "	193.3	9-4	9-11	"
75 "	207.5	9-6	10-1	"
80 "	221.4	9-9	11-4	"
85 "	236.5	10-0	11-7	9-0
90 "	249.7	10-3	11-9	"
95 "	264.7	10-3	11-9	"
100 "	283.0	10-5	11-11	"
110 "	321.0	10-10	12-4	9-6

Erector's Note:
W = Total weight of one single track span

Span	Weight of one girder	Weight of cross frames, etc.
25′ to 60′	.45 W	.10 W
65′ " 110′	.465 W	.07 W

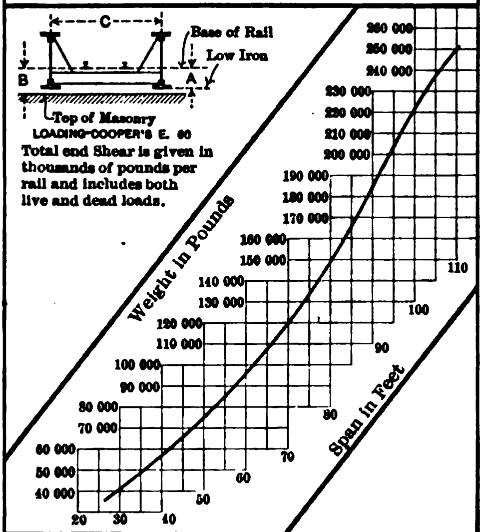
FIG. 5.

may be changed approximately in the ratio of the locomotive weights.

For weights of cantilever spans and nickel steel bridges the reader is referred to the paper by Dr. J. A. L. Waddell entitled

"Nickel Steel for Bridges" published in the Transactions of the American Society of Civil Engineers, Vol. LXIII, pages 165 to 172. For bridges designed for either heavier or lighter loads these weights may be altered in a somewhat less propor-

THROUGH PLATE GIRDER SPANS, I-BEAM FLOOR
WEIGHTS, DIMENSIONS, ETC.



Span	Total End Sh.	A	B	C
30 ft.	106.2	2-2"	2-2"	15-0"
35 "	117.1	"	"	16-0
40 "	128.7	"	"	17-0
45 "	141.1	2-2½	2-2½	17-6
50 "	152.2	"	"	"
55 "	162.1	2-3	2-9	"
60 "	173.2	"	"	"
65 "	186.2	"	"	"
70 "	199.3	2-3½	3-9	"
75 "	212.5	"	3-10½	"
80 "	226.4	"	"	"
85 "	240.5	"	"	"
90 "	257.4	"	"	"
95 "	268.7	"	"	"
100 "	284.0	2-4	"	"
110 "	306.0	2-4½	"	"

Erector's Note:
W= Total weight of one single track span with two light girders.

Span	Weight of one light girder	Weight of one heavy girder	Weight of one floor system
30′ to 50′	.22 W	.39 W	.56 W
55′ " 80′	.27 W	.48 W	.47 W
85′ " 110′	.31 W	.57 W	.38 W

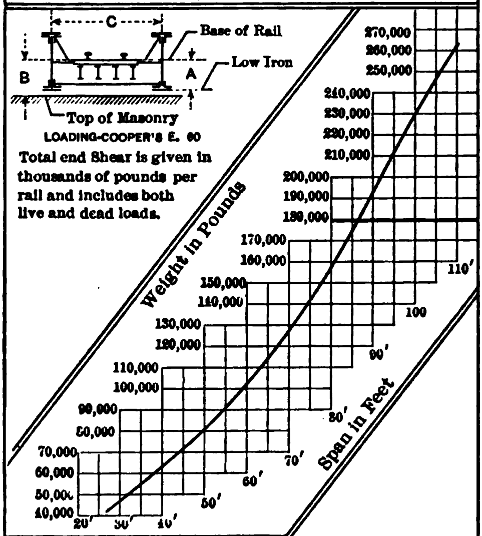
FIG. 6.

tion than the live loads in accordance with the designer's judgment.

6. Approximate Truss Weights. Bridges differing materially from those previously considered and for which other data are not available may be estimated by the following rule devised

by Clarence W. Hudson, Consulting Engineer, 45 Broadway, New York.

THROUGH PLATE GIRDER SPANS, STRINGER FLOOR.
WEIGHTS, DIMENSIONS, ETC.



Span	Total End Sh.	A	B	C
30 Ft.	108.7	3′3″	3′3″	15′6″
35 "	121 0	3′3½″	3′3½″	16′6″
40 "	133.2	"	"	17′6″
45 "	144.7	"	"	"
50 "	156.0	"	"	"
55 "	167.5	3′4″	3′11″	"
60 "	176 3	"	"	"
65 "	191.1	3′4½″	"	"
70 "	204.7	"	"	"
75 "	220.4	"	5′0″	"
80 "	232.8	"	"	"
85 "	248.6	"	"	"
90 "	262.3	3′5″	"	"
95 "	277 4	"	"	"
100 "	291 2	3′6″	"	"
110 "	319 6	"	"	"

Erector's Note.
W = Total weight of one single track span with two light girders.

Span	Weight of one light girder.	Weight of one heavy girder.	Weight of one floor system.
30′ to 50′	.24 W	.42 W	.52 W
55′ to 80′	.25 W	.46 W	.50 W
85′ to 110′	.28 W	.51 W	.44 W

FIG. 7.

Let L = maximum live stress in bottom chord.

I = impact in member in which L occurs.

D_1 = dead stress in same member due to known weight of floor.

D_2 = dead stress in same member due to weight of truss and bracing (guessed).

f_t = allowable unit stress in tension.

Let A_1 = area in square inches of member in which L occurs.
A_2 = area in square inches per linear unit of one truss.
W = weight per linear foot of one truss and its bracing.

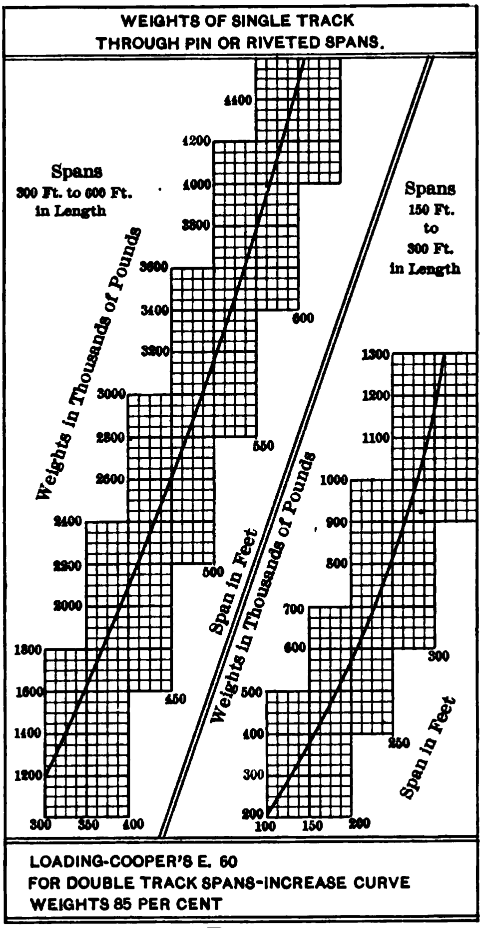


Fig. 8.

Then

$$A_1 = \frac{L + I + D_1 + D_2}{f_t}, \quad A_2 = 5A_1 \quad \text{and} \quad W = \frac{50}{3}A_1. \quad . \quad (1)$$

The above is based upon an allowance of $1.25A_1$ for the upper chord, $1.25A_1$ for the web members, A_1 for details, and $.5A_1$ for bracing. The weight of steel is used in round figures

as 10 lbs. per square inch of cross section for a bar one yard in length.

This method is said to give a very close approximation to the actual weight.

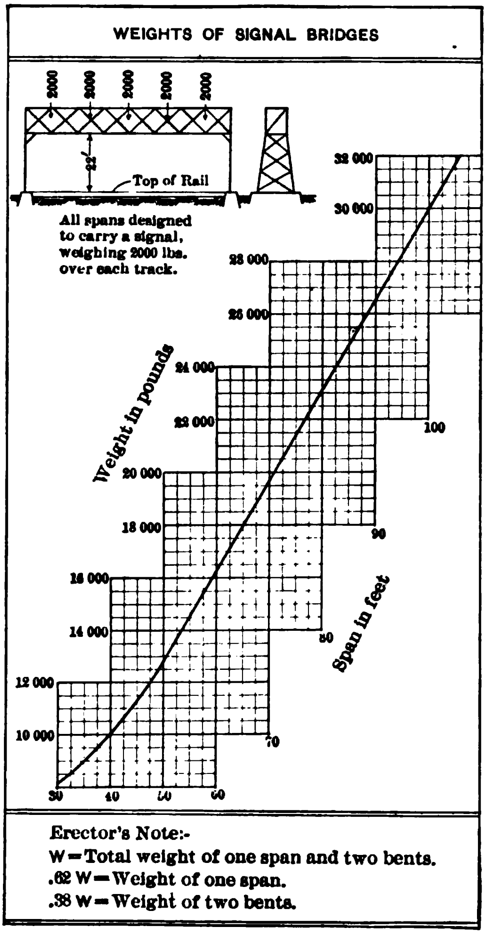


FIG. 9.

7. Weight of Highway Bridges. The weight of highway bridges is less easily determined than that of railroad bridges since the width, span, floor covering and character of loading are subject to wide variations. Formulas have been deduced

for special cases, but these are of little value and will not be quoted. The designer should proceed step by step as previously stated, and if experienced should obtain good results. The method given

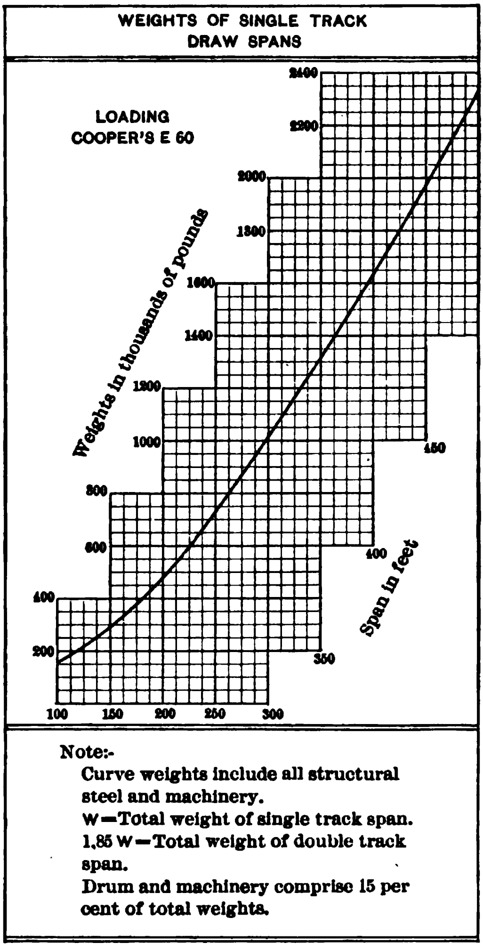


FIG. 10.

in Art. 6 may be applied in order to obtain an approximate truss weight.

The following figures for weights of the steel in actual bridges in the city of Boston may be useful in making estimates for

city bridges. These bridges are all modern structures designed for heavy street car traffic. The floor in all cases consists of yellow pine underplanking, 5 or 6 inches in thickness, waterproofed on top, and supporting a 6-inch stone-block pavement on a sand cushion.

These values together with those of paving materials were furnished the writer by Mr. Frederic H. Fay, formerly Engineer of Bridges and Ferries of the Public Works Department of the city of Boston.

WEIGHT OF STEEL IN TYPICAL HIGHWAY BRIDGES IN THE CITY OF BOSTON

Bridge.	Type.	Span.	Weight of Steel per Square Foot.
Broadway	Plate girder	67 ft. 8½ ins.	52.0 lbs.
Summer Street	"	72 ft. 0 ins. to 85 ft. 0 ins.	55.2 "
Charlestown	"	82 ft. 11 ins.	57.6 "
Craigie (proposed)	"	80 ft. 0 ins.	65.3 "
Northern Avenue	Pin trusses	146 ft. 9⅝ ins.	67.0 "

WEIGHT OF PAVING MATERIAL

NOTE.—B.M. = Board Measure.

Hard (yellow) pine, 4 lbs. per ft. B.M. (Where protected by waterproofing and always dry. Otherwise use 4½ lbs.)	48 lbs. per cu.ft
Creo-resinate yellow pine paving blocks	65 " "
Spruce and white pine, 2½ lbs. per ft. B.M.	30 " "
Bricks, pressed and paving	150 " "
Portland cement concrete	160 " "
Tar concrete (base for asphalt walks, etc.)	125 " "
Silician rock (Simpson Bros.)	140 " "
Trinidad asphalt (Barber Asphalt Co.) refined	74 " "
As laid	140 " "
Granolithic or artificial stone	150 " "
Pavements (exclusive of sand cushion):	
6-inch granite block	80 lbs. per sq.ft.
4-inch brick	50 " "
4-inch wood block (creo-resinate)	22 " "
Roadway waterproofing:	
1½ ins. thick (felt, roofing pitch, sand, and road pitch).	12 lbs. per sq.ft.
Buckle plates	10 to 20 " "

8. Weight of Roof Trusses. The weight of roof trusses depends upon the span, distance apart of trusses, roof covering and roof pitch. The conditions are somewhat more uniform

than for highway bridges, and formulas for approximate weights may be used with some degree of success.

The formula[1] which follows was deduced by N. Clifford Ricker from the weight of 50 roof trusses designed for spans from 20 to 200 ft. in length. The distance apart of trusses varied from 10 to 30 ft. and the rise from $\frac{1}{10}$ to $\frac{1}{4}$ the span. The roofs were assumed as covered with $\frac{7}{8}$ ins. wooden sheathing carrying painted tin. The snow load was taken as 20 lbs. per horizontal square foot and wind as 30 lbs. per square foot on a vertical plane, properly reduced to allow for pitch of roof. Trusses of yellow pine with steel verticals, of white pine with steel verticals, and entirely of steel were included.

Let W = weight of truss in lbs. per square foot of horizontal projection of roof.

S = span in ft.

$$W = \frac{S}{25} + \frac{S^2}{6000}. \quad \dots\dots\dots \quad (2)$$

Steel trusses of shorter spans than 100 ft. probably weigh somewhat more than the value given by the formula, and white pine trusses are somewhat lighter than those of yellow pine.

For roof trusses under other conditions, the value given by the formula may be modified or the designer may use his judgment.

The following table gives the approximate weights per square foot of roof surface for some of the roof coverings in common use (see also latter portion of Art. 14):

	Lbs. per sq. ft.
Pine shingles	2
Corrugated iron, without sheathing	1 to 3.75
Felt and asphalt, without sheathing	2
Felt and gravel, without sheathing	5 to 10
Slate, $\frac{1}{4}$ in. thick	9
Tin, without sheathing	0.75
Skylight glass, $\frac{3}{16}$ to $\frac{1}{2}$ in., including frames	4 to 10
White pine sheathing, $\frac{7}{8}$ in. thick	2
Yellow pine sheathing, $\frac{7}{8}$ in. thick	4
Tiles, flat	15 to 20
Tiles, corrugated	8 to 10
Tiles, on concrete slabs	30 to 35
Plastered ceiling	10 to 15

[1] See "Bulletin No. 16" of the University of Illinois, Engineering Experiment Station, entitled "A Study of Roof Trusses," by N. Clifford Ricker, D.Arch., Professor of Architecture.

9. Weight of Steel-frame Buildings. The weight of such buildings is largely dependent upon the weight of walls, floors, partitions and fire-proofing and these can be estimated in detail from the architect's plans. The weight of the steel is, however, so variable that no attempt will be made to give values for it, but no difficulty need arise in designing, since the weight of the steel, in any given member forms but a very small percentage of the load which it has to carry.

The table which follows may be used in determining the weight of hollow tile floors and walls.

WEIGHTS OF HOLLOW TILE FLOOR ARCHES AND FIREPROOF MATERIALS

Hollow Brick for Flat Arches, Side Construction

Width of Span between Beams.	Depth of Arch.	Weight per Square Foot.
3 feet 6 inches to 4 feet 0 inches..........	6 inches	27 pounds
4 " 0 " 4 " 6 "	7 "	29 "
4 " 6 " 5 " 0 "	8 "	32 "
5 " 6 " 6 " 0 "	9 "	36 "
6 " 0 " 6 " 6 "	10 "	39 "
6 " 6 " 7 " 0 "	12 "	44 "

Partitions

	Thickness.	Weight per Square Foot.
Hollow brick (clay) partitions............	2 inches	11 pounds
" "	3 "	14 "
" "	4 "	15 "
" "	5 "	19 "
" "	6 "	20 "
" "	8 "	27 "
Porous terra-cotta partitions............	3 "	16 "
" "	4 "	19 "
" "	5 "	22 "
" "	6 "	23 "
" "	8 "	33 "

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END CONSTRUCTION, FLAT ARCH

Width of Span between Beams.	Depth of Arch.	Weight per Square Foot.
5 feet to 6 feet	8 inches	27 pounds
6 " 7 "	9 "	29 "
7 " 8 "	10 "	33 "
8 " 9 "	12 "	38 "

FURRING, ROOFING, AND CEILING

	Thickness.	Weight per Square Foot.
Porous terra-cotta furring	2 inches	8 pounds
" roofing	2 "	12 "
" "	3 "	14 "
" "	4 "	18 "
" ceiling	2 "	11 "
" "	3 "	14 "
" "	4 "	18 "

6-inch segmental arches, 26½ pounds per square foot.
8-inch segmental arches, 32 pounds per square foot.
2-inch porous terra-cotta partitions, 8 pounds per square foot.

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Concrete for building work may be made with cinders, broken stone or gravel, and its weight may be taken as follows:

Cinder concrete, 112 lbs. per cubic foot. Trap rock or gravel concrete, 150 to 155 lbs. per cubic foot.

For concrete reinforced with steel add 4 lbs. per cubic foot to above weights.

In practice the minimum weight of a fireproof floor may be taken as 75 lbs. per square foot except for office buildings where 10 lbs. should be added to provide for movable partitions.

Fireproofing for columns or beams may be either of terra-cotta or concrete. The thickness should be not less than two inches. The weight per foot depends upon the size of the member to be protected.

10. Live Loads for Railroad Bridges. It is possible to determine definitely the weights of the locomotives and cars used upon a given railroad. In consequence the actual live loads crossing a given bridge can be ascertained with considerable

exactness, though it is necessary to make due allowance for the effect of high speed, irregularities in track, and other dynamic effects which do not occur when the loads are at rest. These dynamic forces are considered in Arts. 16 and 17 and will be neglected for the present.

In the design for a new bridge it is also desirable to make due allowance for possible increase in weight of locomotives and cars, hence the loads for which bridges are designed may be somewhat heavier than those which are in actual use at the time of construction, though the factor of safety (see Art. 19) provides to some extent for such increase.

As to the type and number of locomotives and character of train loading, American practice is fairly uniform.

Two combinations are usually considered:

(*a*) Two consolidation locomotives followed by a uniform load per foot.

(*b*) A pair of axles with loads somewhat heavier than those of the consolidation engine and no uniform load.

The former loading gives the maximum stresses for most cases, but the latter is sometimes the controlling factor for stringers, short beam spans, and minor truss members.

In designing, the effect of rails and ties in distributing the locomotive load is usually neglected, the wheel loads being considered as applied at points.

As the actual variation in wheel spacing and loads for locomotives of different makes is often slight, it has become in recent years the custom of many railroads to specify the typical locomotives, first proposed by Theodore Cooper, Consulting Engineer, of 45 Broadway, New York. In these locomotives the distance between axles are in even feet and the wheel loads in even thousands of pounds. While these loads and spaces may not represent actual cases they agree closely with average locomotives, and are much simpler to deal with than loadings in odd hundreds of pounds and axle spacings in feet and inches. Moreover, the uncertain developments of the future and the unknown effect of impact make the use of such typical loads but little less accurate than the use of actual wheel loads.

Fig. 11 shows Cooper's E_{60} locomotive, which is suitable for loads carrying heavy traffic. For other conditions types known as E_{50}, E_{40}, E_{30}, etc., are used, these differing from

the E_{60} type in weight only; for example, in the E_{40} type the driving-wheel axle loads are 40,000 lbs. instead of 60,000 lbs., while the other loads are proportionate. This has the advantage of allowing tables of moments and shears made for one type to be readily used for another type by multiplying by a simple ratio. Such tables are now incorporated in numerous handbooks and specifications.

Before leaving the subject a few words as to other methods of loadings are desirable. Some years ago there was considerable agitation in favor of adopting a uniform load, in order to simplify computation. The advantage is obvious to those who are familiar with such work, but the disadvantage is that in order to obtain properly proportioned trusses this load must necessarily vary for different spans and for different members

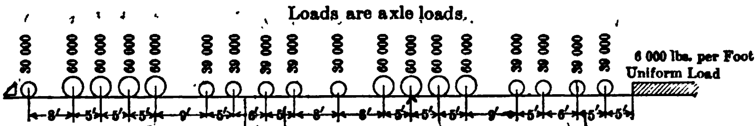

FIG. 11.—Diagram of Cooper's E_{60} Standard Loading.

in the same span. This offsets to a considerable extent the advantage gained. Moreover, the adoption of the above standard loadings has further simplified the labor of computation for actual wheel loads, so that at the present time it is believed that for ordinary structures the advantage in using a uniform load is too small to consider. For complicated trusses a combination of the two methods is perhaps best; viz., the use of wheel concentrations for web members and of a uniform load for the chords, since the approximation for the chords by using a uniform load is less than for the web members.

11. Live Loads for Highway Bridges. The magnitude and character of such loads depend almost entirely upon the location of the bridge. If it be a large city or in a district where heavy manufactured articles or quarried stones of great weight are to be transported it is quite probable that wagon loads from 20 to 30 tons may at times pass over the bridge. Electric-car loads of from 40 to 60 tons should also be assumed, as such cars are already in use in some portions of the United States, while the construction of interurban electric lines in many sections of the country indicates a future widespread extension of heavy

street car traffic. The amount of foot travel and ordinary vehicular traffic on highway bridges also requires careful study. The weight per square foot from a crowd of people may reach the high figure of 150 lbs., which is probably heavier than the weight per foot from horses and wagons on the roadway. To assume, however, that such a load is likely to occur over the entire surface of an ordinary bridge is absurd, and the longer and wider the bridge the less the load that should be taken. In fact, every considerable highway bridge is in itself a problem in loading and should be carefully studied. The following clause from the Massachusetts Public Service Commission's "Specifications for Bridges Carrying Electric Railways," prepared in 1915 by Lewis E. Moore, Bridge and Signal Engineer of the commission, may, however, be used as a broad general guide for the determination of loads for ordinary highway bridges:

"(*a*) For city bridges, subject to heavy loads:

"For the floor and its supports, a uniform load of 100 lbs. per square foot of surface of the roadway and sidewalks, or the auto-truck described under (*d*). In computing the floor beams and supports, the railway load shall be assumed, together with either (1) this uniform load extending up to within 2 ft. of the rails, or (2) the auto-truck described under (*d*).

"For the trusses or girders, 100 lbs. per square foot of floor surface for spans of 100 ft. or less, 80 lbs. for spans of 200 ft. or over, and proportionally for intermediate spans. This uniform load is to be taken as covering the floor up to within two feet of the rails.

"(*b*) For suburban or town bridges, or heavy country highway bridges:

"For the floor and its supports, a uniform load of 100 lbs. per square foot, or the auto-truck described under (*d*); these loads to be used as described under (*a*).

"For the trusses or girders, 80 lbs. per square foot of floor surface for spans of 100 ft. or less, and 60 lbs. for spans of 200 ft. or more, and proportionally for intermediate spans; to be used as described under (*a*). (See *d*).

"(*c*) For light country highway bridges:

"For the floor and its supports, a uniform load of 80 lbs. per square foot or the auto-truck described under (*d*); this load to be used as described under (*a*). (See *d*.)

"For the trusses or girders, 80 lbs. per square foot of floor surface for spans of 75 ft. or less, and 50 lbs. for spans of 200 ft. or more, and proportionally for intermediate spans; to be used as described under (*a*).

" (*d*) All parts of the floor of a highway bridge shall be proportioned to carry a 20-ton auto-truck having 6 tons on one axle and 14 tons on the other axle, the axles being 12 feet apart and the distance between wheels 6 feet. This truck is assumed to occupy a floor space 32 feet in length and 10 feet in breadth, overhanging all wheels an equal amount.

Snow and ice load must also be considered in computing the stresses in draw spans when open since such stresses may attain considerable importance. The magnitude of these loads in the vicinity of New York will probably not exceed 10 lbs. per square foot. For fixed span bridges snow need not be taken into account since the maximum wagon and other loads will not occur simultaneously with the snow load.

12. Live Loads for Buildings. The proper loads for buildings depend upon the purpose for which the building is to be used, and in the larger cities is generally prescribed by law.

The live loads which follow are the minimum live loads prescribed by the present New York city building laws and represent good practice:

FLOORS

Dwelling-house, apartment house, or hotel	60	lbs. per sq.ft.
Office building, first floor	150	"
Office building, other floors	75	"
School house	75	"
Stable or carriage house	75	"
Place of public assembly	90	"
Ordinary stores, light manufacturing, or light storage	120	"
Stores where heavy materials are kept, warehouses, and factories	150	"

ROOFS

All roofs with pitch less than 20°, 50 lbs. per sq.ft. of surface.
All roofs with pitch more than 20°, 30 lbs. per sq.ft. of horizontal projection of surface.

" For columns of dwellings, office buildings, stores, stables, and public buildings when over five stories in height a reduction of the live loads may be made as follows:

" For roof and top floor use full live load; for each succeeding

lower floor reduce live load by 5 per cent, until 50 per cent of the live loads fixed by this section is reached, when such reduction or such reduced loads shall be used for all remaining floors."

For further information upon live loads for buildings the student is referred to the article by C. C. Schneider in the Transactions of the American Society of Civil Engineers, Vol. LIV, page 371 et seq., with the ensuing discussion.

13. Wind Pressure. Wind pressure is a subject upon which little exact information exists, although many experiments have been made and much study given to the subject by engineers and scientists. Among the unsettled questions are:

a. The relation between pressure and velocity.

b. The variation of pressure with size and shape of exposed plane surfaces.

c. The direction and intensity of pressure upon non-vertical surfaces.

d. The intensity of pressure upon non-planar surfaces.

e. The total pressure upon a number of parallel bars or other members placed side by side.

f. The decrease of pressure upon leeward surfaces.

g. The lifting powers of the wind.

a. In comment upon this subject it may be said that the pressure varies about as the square of the velocity and that the results given by different experimenters vary from

$$P = .005V^2 \qquad \text{to} \qquad P = .0032V^2,$$

of which the latter value represents the result of unusually careful experiments by Stanton[1] upon the intensity of pressure on plates varying in size from 25 to 100 sq.ft. and is probably more nearly correct than the higher value.In these formulas

[2] P = pressure in pounds per square foot,
[3] V = velocity in miles per hour.

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[1] See Minutes of Proceedings of the Institute of Civil Engineers, Vols. CLVI and CLXXI.

[2] Pressure specified for bridge in Philippines recently was as high as 160 lbs. per sq.ft., according to Howard C. Baird, of Boller, Hodge & Baird, Consulting Engineers, New York City.

[3] Wind velocity in New York City on Feb. 22, 1912, reported as 120 miles per hour.

In the Stanton formula the values are reduced to correspond to a temperature of 60° F. and an atmospheric pressure of 14.7 lbs. per square inch.

b. The variation of pressure with size and shape of exposed surface is important and is not well understood, although it is sure that the resultant pressure on a large surface may be taken as less per square foot than that on a small surface, since the maximum intensity of the wind is due to gusts of comparatively small cross-section.

c. The pressure upon vertical plane surfaces may be taken as normal to the surface and equal in intensity to the assumed wind pressure. Upon surfaces which are not vertical, the pressure is usually considered to be normal to the surface but lower in intensity than upon vertical surfaces. The variation in pressure with respect to the slope is not well understood and a number of empirical formulas are in use, among which may be noted the much used Duchemin formula

$$P_n = P\frac{2 \sin i}{1+\sin^2 i}, \quad \ldots \ldots \quad (3)$$

and the Hutton formula

$$P_n = P(\sin i)^{(1.84 \cos i - 1)}, \quad \ldots \ldots \quad (4)$$

in which P_n = intensity of normal pressure upon the given surface,
P = intensity of normal pressure upon the vertical surface.
i = angle made by surface with the horizontal.

A formula may be deduced on the assumption that the wind always blows in horizontal lines, and that if the pressure be resolved into normal and tangential components, the tangential component may be neglected.

The demonstration is as follows:

Let the wind be assumed as blowing horizontally against the surface *ab* of height *bc* and making an angle *i* with the horizontal.

Let the length of the surface be one foot, perpendicular to the plane of the paper. (See Fig. 12.)

Let P = intensity per square foot of the horizontal wind force on a vertical surface.

P_n = intensity per square foot of the normal force acting on surface ab.

P_t = intensity per square foot of the tangential force acting on surface ab.

The total horizontal pressure on surface ab then equals Ph.

The normal component of this pressure $= Ph \sin i$.

The intensity of the normal component $= \dfrac{Ph \sin i}{ab}$.

But $ab = \dfrac{h}{\sin i}$, $\therefore\ P_n = P \sin^2 i$. (5)

This formula gives lower values than the empirical formulas (3) and (4) and probably gives too low results since it makes no allowance for the reduction in pressure on the leeward side which is known to exist, and which may in part be attributed to the influence of the tangential component. It should also be noted that the wind does not blow uniformly in horizontal lines but may deviate considerably from this direction.

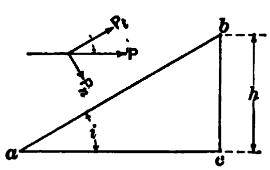


FIG. 12.

The values given by these three formulas are tabulated for comparison on page 23, using an assumed value of 30 lbs. per square foot for P.

In the absence of further experience upon this phase of wind pressure it would seem wise to use one of the empirical formulas instead of the theoretical one, and the Hutton formula (4) is frequently used by structural engineers in England and the United States.

The following theorem relating to the wind pressure upon plane surfaces is particularly useful in determining reactions upon roof trusses:

The *horizontal* component of the total normal pressure upon a plane surface equals the intensity of the normal pressure multiplied by the area of the *vertical* projection of the surface, and the *vertical* component of the total normal pressure equals that intensity multiplied by the area of the *horizontal* projection of the surface.

This theorem applies to any surface subjected to a uniformly distributed *normal* pressure and may be proven as follows:

Let P_n = intensity of the normal force acting upon surface ab.
P_h = horizontal component of total normal force upon ab.
P_v = vertical component of total normal force upon ab.
bc = vertical projection of ab.
ac = horizontal projection of ab.
θ = angle between ab and horizontal.

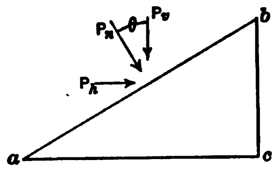


FIG. 13.

Assume surface ab to be of length unity perpendicular to paper.

Then total normal pressure on $ab = P_n \times ab$.

hence $$P_h = P_n \times ab \times \sin\theta = P_n \times ab \times \frac{bc}{ab} = P_n \times bc$$

and $$P_v = P_n \times ab \times \cos\theta = P_n \times ab \times \frac{ac}{ab} = P_n \times ac.$$

Since bc and ac are the vertical and horizontal projections of ab, the theorem is proven.

TABLE FOR WIND PRESSURE. P = 30 LBS. PER SQ.FT.

i	$P \sin^2 i$	$P\frac{2\sin i}{1+\sin^2 i}$ Duchemin Formula.	$P(\sin i)^{(1.84\cos i - 1)}$ Hutton Formula.
5°	0.0	5.2	3.9
10°	0.9	10.1	7.3
15°	2.0	14.6	10.5
20°	3.5	18.4	13.7
25°	5.3	21.5	16.9
30°	7.5	24.0	19.9
35°	9.9	25.8	22.6
40°	12.4	27.3	25.1
45°	15.0	28.3	27.0
50°	17.6	29.0	28.6
55°	20.1	29.4	29.7
60°	22.5	....	....
65°	24.6	Above 60°	Above 60°
70°	26.4	use 30 lbs.	use 30 lbs.
75°	28.0		
80°	29.1		
85°	29.7		
90°	30.0		

It will be observed that if θ is greater than 45°, P_h will be larger than P_v; if less P_v will be the larger. It is obvious that this is correct since the steeper the roof the greater the horizontal component. When $\theta=90°$, $P_v=0$ and when $\theta=0$, $P_h=0$.

The application of this method to the solution of a problem is given in Art. 25.

d. The pressure upon non-planar surfaces is important in the case of chimneys, stand-pipes, and other similar objects.

If the assumptions previously made in the deduction of formula (5) be also made for curved surfaces the total pressure upon such surfaces can be easily figured. The following demonstration shows the solution for the case of a vertical cylinder.

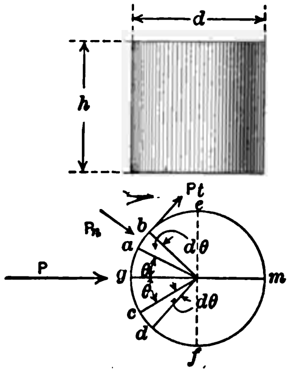


FIG. 14.

Let P = intensity of pressure on a vertical plane.

P_n = intensity of pressure on a plane making an angle of $90°-\theta$ with the direction of the wind $=P\sin^2(90°-\theta)$.

P_t = tangential component of pressure on same plane.

The normal pressure on the differential area ab subtended by the angle $d\theta = P\sin^2(90°-\theta)\frac{hd}{2}d\theta$.

As the tangential component P_t is neglected by hypothesis and the component of P_n acting upon surface ab in a direction parallel to ef is balanced by an equal and opposite component upon cd, the force tending to overturn the cylinder is the summation of the components of P_n parallel to gm. and is given by the following expression:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P\sin^2(90°-\theta)h\frac{d}{2}d\theta\cos\theta = \frac{Phd}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta d\theta$$

$$=\frac{Phd}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,(1-\sin^2\theta)d\theta=\frac{Phd}{2}\left(\frac{4}{3}\right)=\frac{2}{3}Phd$$

= two-thirds of the total pressure on a plane diametrical section.

In a similar manner the pressure on a spherical or conical surface may be computed.

The pressures obtained by this method lack experimental proof but are probably more nearly correct than the pressure obtained by the same method upon plane surfaces. The value given for the cylinder is quite generally used.

e. With respect to the total pressure upon a number of parallel bars placed side by side it may be stated that experiments previously referred to indicate that the total pressure on a pair of circular plates placed 1½ diameters apart is less than that on one plate from which the conclusion is drawn, that the pressure on the leeward plate is in a direction opposite to current. When plates were placed 2.15 diameters apart the resultant pressure on the two plates was found to equal that on a single plate and the shielding effect was found to be well maintained with wider spacing, since at a distance of five diameters the total pressure was only 1.78 that on a single plate.

f. The pressure upon the windward side of an exposed surface is a function of the density and velocity of the air currents. The pressure on the leeward side is also a function of the shape of the surface, and has been shown by numerous experiments to be less than the static pressure of the air current. The resultant total pressure upon a surface is in consequence not only a function of the direct pressure on the windward side but also of the pressure on the leeward side, which in turn is a function of the form of the surface. It is therefore doubtful if an algebraical formula can be deduced which will give the pressure on surfaces of varying shape with any considerable degree of precision.

g. In the case of a very rapid reduction of atmospheric pressure, as in a tornado, it is often observed that building roofs are lifted and walls blown outward. This phenomenon is due to the air in the building, which is under more or less restraint, changing pressure less rapidly than the outside air and thereby producing a difference in pressure. This lifting action doubtless occurs to a greater or less degree whenever the external pressure is reduced, and should be guarded against by anchoring roofs securely to the walls.

The many uncertainties connected with wind pressure make worthless the attempts to specify with precision its magnitude and direction. In the lack of additional information and fur-

ther theoretical studies there seems to be no reason for deviating from the common rules which have been used in bridge design in the United States for many years with satisfactory results. These rules may be stated as follows:

The portal, vertical and horizontal bracing shall be proportioned for a wind pressure of 30 lbs. per square foot on the surface of the applied load, and on the exposed surfaces of the floor system and both trusses. The pressure on the applied load shall be considered as a moving live load, and the other pressure as a dead load. For structures of ordinary spans the wind stresses shall also be computed upon the unloaded structure for a pressure of 50 lbs. per square foot. In the design the maximum stress computed by either of the above methods shall be used.

The wind stresses in main truss members shall also be computed, but if the combined stress in any members due to dead load, vertical live load and wind load does not exceed by more than 20 per cent the allowable unit stress no allowance in the main members need be made for the wind.

In computing the area of exposed surface take twice the front surface of members composed of many bars, and 1.5 that of bars in pairs. The pressure upon the ends of ties, and upon the guard timbers should not be neglected and may be considered as one square foot per linear foot of bridge.

For wind pressure on roofs and buildings it is common practice to allow 30 lbs. per square foot acting horizontally upon the sides and ends of buildings, or on the vertical projection of roofs. It is also very important to figure the wind stresses on the steel frame considering it as an independent structure without walls, floors or partitions, since failures often occur in erection.

For lateral pressure on steam railroad bridges and trestles, due account should be taken of the sidewise vibration of the train in addition to the wind force. The following paragraphs from the general specifications of the American Railway Engineering and Maintenance of Way Association for steel railroad bridges may be used as a guide.

All spans shall be designed for a lateral force on the loaded chord of 200 lbs. per linear foot plus 10 per cent of the specified train load on one track, and 200 lbs. per linear foot

on the unloaded chord; these forces being considered as moving.

Viaduct towers shall be designed for a force of 50 lbs. per square foot on one and one-half times the vertical projection of the structure unloaded; or 30 lbs. per square foot on the same surface plus 400 lbs. per linear foot of structure applied 7 ft. above the rail for assumed wind force on train when the structure is either fully loaded or loaded on either track with empty cars assumed to weigh 1200 lbs. per linear foot, whichever gives the larger strain.

14. Snow Load. The weight per foot of snow and ice varies greatly with climatic conditions. The following rule suggested by C. C. Schneider in the paper recently referred to gives reasonable results for conditions similar to those existing in Boston and New York:

" Use for all slopes up to 20° with the horizontal 25 lbs. per square foot of horizontal projection of roof. Reduce this value by one pound for each additional degree of slope up to 45°, above which no snow need be considered."

To determine the maximum stresses in a truss member, wind and snow must be properly combined. The following combinations may exist and should be considered:

Dead load with snow on both sides.

Dead load with snow on one side and wind on the other.

Dead load with ice at 10 lbs. per square foot, properly reduced according to slope, on both sides, and wind on one side.

The maximum stress as determined by either of these combinations should be used.

For roof trusses of short span it is becoming the custom to combine the snow, wind, and dead load by using a value sufficient to cover them all. The following are suggested by Schneider as minimum loads per square foot of exposed surface to provide for combined dead, wind and snow loads on spans less than 80 ft. These loads are to be taken as acting vertically.

	Lbs.
Gravel or composition, on boards, flat slopes, 1 to 6 or less	50
Gravel or composition, on boards, steep slopes, more than 1 to 6	45
Gravel or composition, on 3-inch flat tile or cinder concrete	60
Corrugated sheeting on boards and purlins	40
Slate on boards and purlins	50
Slate on 3-inch flat tile or cinder concrete	65
Tile on steel purlins	55

Where no snow is likely to occur these values are to be reduced by 10 lbs., but no roof is to be designed for less than 40 lbs.

For roofs with other coverings than those above use 30 lbs. per square foot of horizontal projection for combined effect of snow and wind on all slopes.

15. Centrifugal Force and Friction. For railroad bridges on curves the effect of centrifugal force must be considered. This may be computed by the following formula:

$$C = 0.03WD \text{ for a curvature up to } 5°, \quad . \quad . \quad . \quad (6)$$

where C = centrifugal force in pounds;
W = weight of train in pounds;
D = degree of curvature.

The coefficient for centrifugal force (0.03) to be reduced 0.001 for every degree of curvature above 5°.

On trestle towers and similar structures the longitudinal thrust of the train must be considered. This may be taken as having its maximum value when the brakes are set and the wheels sliding, and may be computed assuming a value of 0.2 for the coefficient of sliding friction.

16. Impact on Railroad Bridges. It is easy to determine the weight per wheel applied to a railroad bridge by the locomotive or cars of a given train when at rest, but when in motion the effect of unbalanced locomotive drivers, roughness of track, flat, irregular or eccentric wheels, rapidity of application, and centrifugal force induced by deflection of structure, cannot be determined theoretically and has not yet been precisely determined by experiment. Neither is the distribution of these loads by rails and ties a matter which can be easily ascertained. In consequence the engineer is compelled either to use a low unit stress or to increase the live stresses by an allowance for "impact" sufficient to cover these uncertainties. The latter method is more scientific and is coming into general use. Impact is used in mechanics to mean the dynamic effect of a suddenly applied load, but as used in bridge engineering it stands for the increased stress produced in a member not only by the rapid application of the load, but also by the other causes just mentioned, and the term "coefficient of impact" is given to the factor by which the live stress must be multiplied to obtain the impact. No

rational formula for determining this coefficient of impact has yet been deduced, but several empirical formulas are in more or less common use.

It is proven in mechanics that a load when instantaneously applied to a bar produces a stress exactly double that caused by the same load when gradually applied. In the ordinary structure the maximum load is, however, never applied instantaneously, though in short railroad bridges the length of time required to produce maximum moment or shear is very small. In consequence sudden application alone is never sufficient to double the live stresses as computed for quiescent loads. Many engineers, however, use for short spans a coefficient equal to unity, assuming that the effect of vibration and other uncertainties is balanced by the difference between the stress due to instantaneous application and that due to the very rapid but not instantaneous application caused by a railroad train. For longer spans the coefficient is generally reduced.

The two following formulas represent two different types of impact formulas:

From Specifications of the American Railway Engineering and Maintenance of Way Association,

$$I=S\frac{300}{L+300}. \qquad (7)$$

From Specifications of the Department of Railroads and Canals, Canada,

$$I=S\frac{S}{S+D}. \qquad (8)$$

In these formulas

I = impact.

L = length in feet of distance which must be loaded to produce maximum live stress in member.

S = that maximum live stress, and

D = the dead stress.

It will be seen at once that for short spans the coefficient in the first formula is very nearly unity, and that this also applies to the second formula unless the bridge is unusually heavy or the live loads very light. The first formula has been quite generally adopted by American engineers, and while purely empirical agrees reasonably well with such experiments as have been

made, and is in a logical form. It will be used hereafter in this book.

For full discussion of impact upon railroad bridges with experimental data, see Bulletin No. 125 of the American Railway Engineering and Maintenance of Way Association.

17. Impact on Highway Bridges and Buildings. Allowance for impact upon these structures may usually be less than for railroad bridges. The Massachusetts Public Service Commission's Specifications specify 50 per cent allowance for impact due to auto-truck described on page 19 for steel stringers, floor beams, hangers, and truss members receiving their whole load from one panel point only. For other live loads these specifications provide for 25 per cent impact for steel floor beams and stringers, 40 per cent for floor-beam hangers and counters, and a uniformly varying factor for main girder and other truss members varying from 25 per cent for a 20 ft. span to 10 per cent for 200 ft. and longer spans.

For buildings it is customary to make no allowance for impact, except where moving cranes or other shock-producing machinery are used.

Before leaving the subject of impact it should be noted that it is probable that the effect of impact upon wooden beams is less injurious than upon steel beams owing to the greater elasticity of the wood, and that unit stresses for timber as generally specified are for use without impact.

18. Inner Forces. The allowable working unit stresses for a given structure depend upon the material, the character of loading, the precision with which the stresses can be computed, and the uses to which the structure is to be put. If proper allowance for impact be made the character of loading may, however, be neglected.

The following unit stresses represent good practice for ordinary structural steel structures, provided *proper allowance for impact* be made.

WORKING STRESSES. STRUCTURAL STEEL

Ultimate tensile strength from 56,000 to 64,000 lbs. per sq. inch

LIVE LOAD TO BE INCREASED TO ALLOW FOR IMPACT

Tension on net section, and extreme fibre stress in bending	16,000 lbs. per sq.in.
Compression in columns	$16{,}000-70\frac{l}{r}$ lbs. per sq.in. with a maximum of 14,000 lbs.
Shear on net section of plate girder webs and on machine-driven shop rivets	12,000 lbs. per sq.in.
Bending on extreme fiber of pins	24,000 "
Bearing on pins and shop-driven rivets	24,000 "
Bearing on hand-driven rivets	18,000 "
Shear on hand-driven rivets	9,000 "
Modulus of elasticity	28,000,000

These unit stresses may be increased by 25 per cent in case total stress in member is found by combining live, dead, impact and wind stresses.

In the expression for the compression in columns $\frac{l}{r}$ = maximum value of ratio of the unsupported length of column to radius of gyration, both values being expressed in inches. This ratio should be restricted by the form of the column so that it will not exceed 100 for main members and 120 for lateral and other secondary members.

The following values, with the exception of that for tension, are recommended for timber for railroad bridges by the American Railway Engineering and Maintenance of Way Association, and may be used for green timber and without allowance for impact. For highway bridges and trestles these figures may be increased by 25 per cent, and for buildings when protected from weather and reasonably free from impact by 50 per cent. For these values and for timber other than yellow pine see "Proceedings of the American Railway Engineering and Maintenance of Way Association," Vol. 10, Part I, p. 564.

WORKING STRESSES. LONG-LEAF YELLOW PINE

NO ALLOWANCE FOR IMPACT REQUIRED

Bearing along grain	1,300 lbs. per sq.in.
Compression in columns, length over 15 diameters	$1{,}300\left(1-\frac{l}{60d}\right)$
Tension parallel to grain	1,400 lbs. per sq.in.
Bending, extreme fibre stress	1,300 "
Shearing along grain in beams	120 "
Shearing along grain in chord blocks, etc	180 "
Bearing across grain	260 "
Modulus of elasticity	1,600,000 "

$\frac{l}{d}$ = maximum value of ratio of unsupported length of column to least diameter, both values being expressed in inches.

For deflection of yellow-pine beams under long-continued loading, use for modulus of elasticity 800,000 lbs.

The following values for bearing on masonry represent good practice:

WORKING STRESSES. BEARING ON MASONRY

LIVE LOAD TO BE INCREASED TO ALLOW FOR IMPACT

Granite masonry and Portland cement concrete.....	600 lbs. per sq.in.
Sandstone and limestone........................	400 "

19. Factor of Safety. The unit stresses given in the previous article are all much less than the breaking strength of the material, the ratio between the breaking strength and the allowable unit stresses being known as the "factor of safety." The necessity for using such a low value is due to the following facts:

1. Material can not be stressed with safety above the elastic limit, which is generally not more than one half the breaking strength.

2. The magnitude, point of application, and distribution of the live loads as well as the allowance for impact is approximate.

3. The material is variable in quality, and may be injured in fabrication.

4. The effect of changing conditions can not be predicted. This applies to character and amount of loading and to deterioration of material through rust or rot.

5. The common theories give primary stresses only and neglect the secondary stresses due to distortion of the structure, these additional stresses being sometimes of considerable importance.

PROBLEMS

1. Using the Hutton formula, determine the horizontal and vertical components of the total wind force on the side L_0U_2 of roof truss A, for an assumed wind pressure of 30 lbs. per square foot on a vertical surface. Direction of wind shown by arrows.

2. Determine the horizontal wind pressure per square ft. required to tip the car, assuming the direction of the wind to be perpendicular to its side. Neglect wind pressure on trucks.

3. Compute the impact in pounds by formulas (7) and (8) for a bridge member subjected to the following conditions:

Maximum live stress......................	200,000 lbs.
Dead stress...............................	100,000 "
Loaded length when stress is a maximum....	100 ft.

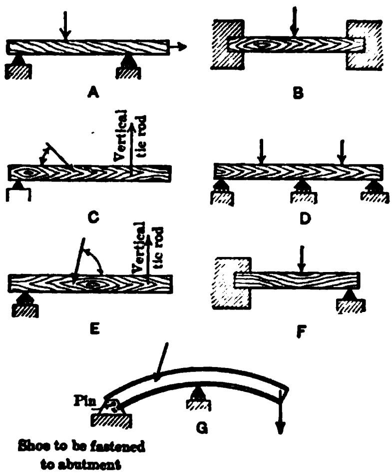
4. State whether each of the beams shown is statically determined with respect to the outer forces and give reasons. Assume that the magnitude and position of applied loads, and position of points of support are known in all cases.[1]

4a. Estimate from the diagram of Fig. 8 the total weight of steel in a double-track through pin bridge of 150 ft. span. and determine its total cost, assuming the steel to cost four cents per pound, erected and painted.

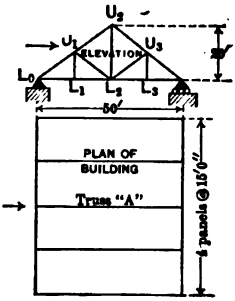
4b. Determine weight per lineal foot of track of a railroad bridge floor made up as follows:

Ties, yellow pine, 8″ × 8″ × 10′–0″–12″ c. to c. Guard timber, yellow pine, 8″ × 6″. Rails and fastenings, 150 lbs. per lineal ft. of track.

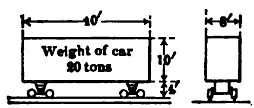
4c. Determine weight of ballast per lineal foot of a single-track solid floor bridge assuming average depth and width of ballast to be 14 in. and 13 ft. respectively. Weight of ballast per cu. ft. to be assumed as 100 lbs.



PROB. 4.



PROB. 1.



PROB. 2. Standard Gauge.

[1] Read Articles 20 and 21 before solving this problem.

CHAPTER II

LAWS OF STATICS, REACTIONS, SHEARS AND MOMENTS, INFLUENCE LINES

20. Laws of Statics. The theory of structures is based upon the fundamental principles of statics, and these the student must thoroughly understand.

For the present *structures in a plane and with the applied loads acting in the same plane alone will be considered.* Such structures will be in equilibrium if the following conditions are satisfied:

1. The algebraic sum of the components of all the forces acting parallel to any axis in the plane of the forces must equal zero.

2. The algebraic sum of the moments of all the forces about any axis at right angles to the plane of the forces must equal zero.

If the forces be resolved into components parallel to two rectangular axes, OX and OY, and the algebraic sum of the forces parallel to OX be designated as ΣX and of these parallel to OY as ΣY, the first of above conditions will be fulfilled when $\Sigma X=O$ and $\Sigma Y=O$, hence the two principles stated above are fully comprehended by the three following equations:

$$\Sigma X=O, \quad \Sigma Y=O, \quad \Sigma M=O.$$

If the forces acting upon a body do not satisfy all of these three equations, then the body cannot be in equilibrium. For example, if $\Sigma X=O$ and $\Sigma Y=O$, but ΣM does not, the body is in a condition of rotation about a stationary axis. If $\Sigma Y=O$ and $\Sigma M=O$, but ΣX does not, then the body has a motion of translation in a direction parallel to the X axis but no other motion.

It is often advantageous to use $\Sigma M=O$ more than once, using different axes. If used three times without the other equations, it will satisfy both $\Sigma X=O$ and $\Sigma Y=O$ as well as $\Sigma M=O$.

In practice it is common to use horizontal and vertical axes for which case the first two equations may be written

$$\Sigma H=O \quad \text{and} \quad \Sigma V=O.$$

21. Reactions. Each of the reactions upon a structure may have three unknown properties, viz., magnitude, direction, and point of application. Usually, however, the point of application of each reaction is fixed in position and the direction of at least one of the reactions is known. If this condition exists when there are two points of support, i.e., two reactions, as is the case in most structures, there remain but three unknown properties of the reactions, all of which may be computed by the three equations of statics, and the structure is statically determined with respect

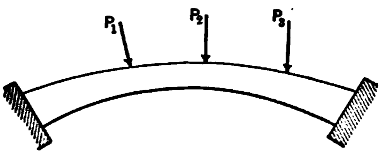


FIG. 15.

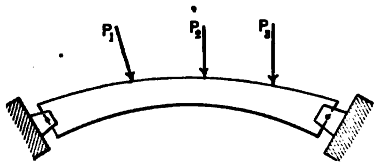


FIG. 16.

to the outer forces, whether it is or is not possible to determine the inner stresses by statics. If there are more than three unknown properties of the reactions, e.g., if only the points of application are fixed; or if the structure is supported on more than two points, then it is statically undetermined with respect to the outer forces, unless some special form of construction is adopted, as in the three-hinged arches and cantilever bridges considered later. If there are fewer than three unknowns, then

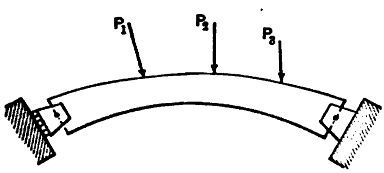


FIG. 17.

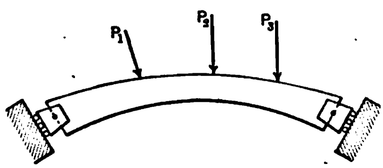


FIG. 18.

the structure is in general unstable and will tend to move bodily under the applied loads unless these fulfil certain special conditions.

Illustrations of the above conditions are afforded by the structures shown in Figs. 15, 16, 17 and 18, for all of which the position and magnitude of the applied loads, and all of the dimensions of the structure are supposed to be known.

Fig. 15 represents the ordinary masonry arch in which each reaction is unknown in direction, magnitude, and point of application. In consequence the structure is indeterminate with respect to the outer forces in a three-fold degree. Fig. 16 shows a two-hinged arch which has the point of application of both reactions determined, but the magnitude and direction of neither are known, hence it is indeterminate in the first degree with respect to the outer forces. In Fig. 17 a set of rollers is shown at one end. The function of these rollers is to make the reaction at that end perpendicular to the supporting surface since the rollers, if in good condition, can offer but little resistance to motion along this surface. This structure is, therefore, statically determined with respect to the outer forces since the points of application of both reactions and the direction of one are known. In Fig. 18 rollers are shown at both ends, hence the direction of both reactions are known. Unless these reactions meet on the line of action of the resultant of the applied loads, equilibrium can not exist and the structure will move, therefore the structure is unstable.

22. Computation of Reactions—Method of Procedure. It is evident that if the horizontal and vertical components of a reaction which is unknown in direction and magnitude, or if either component of a reaction which is known in direction but not in magnitude be determined, the reaction itself may be at once obtained. In consequence the determination of the reactions in a structure which is statically determined with respect to the outer forces and hence has but three unknowns, may be accomplished by computing the horizontal and vertical components of one reaction and either component of the other.

This method often, though not always, simplifies the solution of reaction problems and will be used hereafter. Its adoption makes it desirable to use the horizontal and vertical components of the outer forces and these also can frequently be computed more easily than the actual forces. With these components of the outer forces known the solution of the problem may be accomplished by the application of the three statical equations.

The following mode of procedure is suggested for the use of the beginner, who is advised to follow it exactly until he has mastered the method thoroughly. For structures in which the reactions are not parallel to the forces or in which the character

of the unknown reactions can not be easily predicted, even the experienced computor should not omit any of the steps in the process:

1. Draw a careful sketch of the structure and show on it the horizontal and vertical components of the outer forces. This sketch need not be to scale but should not be materially distorted.

2. Indicate on the sketch by arrows, and by the letters H and V, the assumed components of the reactions, using letters R and L as suffixes of H and V to indicate right and left reactions.

The direction of the components of the reactions which are unknown in direction may be assumed at random, e.g., the horizontal component may be assumed as acting either to the right or the left and the vertical component either up or down, but the components of the reaction the direction of which is known must be so assumed as to be consistent with this known direction.

3. Determine the unknown H and V components by the solution of the equations $\Sigma H=O$, $\Sigma V=O$, and $\Sigma M=O$, considering as positive, forces acting upwards or to the right and clockwise moments.

A positive result shows that the component in question acts in the direction *originally assumed*, and not necessarily that it acts up or to the right. With the magnitude of all components known, the magnitude of either reaction may be obtained by computing the square root of the sums of the squares of its two components. Its direction is determined by the direction of the components. The beginner is more likely to make errors by omitting some of the forces than in any other way. Particular attention may well be called to the fact that horizontal forces may produce vertical reactions and *vice versa*.

If the load, or any portion of it, be distributed over a considerable distance instead of being applied at a point, the resultant of this portion of the load may ordinarily be used in the computations as a concentrated load. This method, however, should be used only in reaction computations; it would in general be incorrect for the determination of shears, moments, and truss stresses. It is also incorrect for the determination of reactions in three-hinged arches.

It is always desirable to obtain a check by twice applying the equation, $\Sigma M=O$, once about each point of support. This

gives an independent check for at least one of the reaction components, which in the case of a simple beam with vertical loads is sufficient and conclusive.

23. Reaction Conventions. Hereafter, in both text and problems, structures supported at one end upon a set of rollers or by a tie-rod will be considered as having the reaction at that point fixed in direction. The reasons for this in the case of rollers is stated in Art. 21. For the tie-rod, it is sufficient to recall that such a rod is little better than a stiff rope and is incapable of carrying bending or compression, hence the reaction which it carries must act along its axis and produce tension in the rod.

Rollers will be indicated by this conventional symbol, and the reaction in this case is always to be considered as perpendicular to the supporting surface, whether the surface be horizontal, inclined or vertical.

When the point of application of a reaction is fixed but not its direction this symbol, will be used. This is not intended to represent a knife edge bearing since the reaction may act in any direction, i.e., up, down, horizontal or inclined. If this symbol be combined with rollers, then both point of application and direction of reaction are to be considered as fixed. If the reaction be carried by a tie-rod, the rod will be so marked; in this case the point of application should be taken at the point where the rod is fastened to the structure.

24. Point of Application of Loads and Reactions. In practice it is seldom that the point of application of load or reaction is definitely fixed; it is, however, in many cases fixed within such small limits that no error arises in considering it as located at a definite point. This is the case when the structure is supported on steel pins, as in most bridges of considerable size; the reaction in such a case passes through the pin, which is generally but a few inches in diameter and its resultant will pass through the pin centre, or nearly so, unless the pin be badly turned or the bearing surface upon which it rests imperfect. With wheel loads the load acts at the point of tangency of wheel and bearing surface, which is practically a point, but as the wheel does not rest directly on the structure but has its load distributed by rails and ties, or by the floor if a highway bridge, it is not applied to the struc-

ture itself at a point, though it is generally so considered as the error thus arising is small and on the safe side.

For ordinary beams which rest at the ends upon steel-bearing plates inserted to distribute the load over the masonry supports, the assumption that the reaction is applied at the centre of bearing is by no means an exact one. The actual distribution of the reaction in such a case is a function of the relative elasticity of the beam and support. If both beam and support were to be absolutely rigid—an impossible case—the reaction would pass through the centre of bearing; if the support alone were to be rigid the reaction would pass through the edge of the bearing plate; in the actual case where both beam and bearing surface yield to some extent, the reaction is distributed over the entire surface and its intensity varies uniformly or nearly so, as shown in Fig. 19. It will be noticed that the resultant pressure acts at a point between the centre of bearing and inner edge of the masonry. The common assumption for such cases is to assume the reaction as applied at the centre of bearing. This assumption is on the *safe* side in designing the beam as a whole, but on the *unsafe* side in proportioning the area of bearing. However, the error for short beams which deflect but little, is not serious. For long girders which deflect considerably the end bearing is usually made by a pin which is supported upon a shoe which in turn rests upon rollers, thus ensuring a uniform distribution of the reaction.

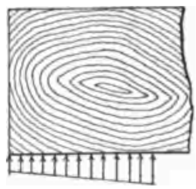


FIG. 19.

25. Solution of Reaction Problems. The application of the methods of Art. 22 is illustrated in the problems of this article.

Problem. Compute horizontal and vertical components of reactions on beam shown in Fig. 20. Neglect weight of beam itself.

Solution. First apply $\Sigma H = 0$. This gives the equation $H_L = 0$, since the applied loads are all vertical and in consequence have no horizontal components.

Now apply $\Sigma M = 0$, taking for origin of moments the point of application of either reaction, thus eliminating one unknown. The equation which follows is derived by taking moments about the right end.

$$-10{,}000 \times 26 + 16V_L - 5000 \times 12 - 10{,}000 \times 2 = 0.$$

$$\therefore\ 16V_L = 340{,}000 \text{ ft.-lbs.} \quad \text{and} \quad V_L = +21{,}250 \text{ lbs.}$$

Since the value of V_L is positive the reaction acts in the direction assumed in the figure.

The application of $\Sigma V=0$, using the value of V_L just obtained,

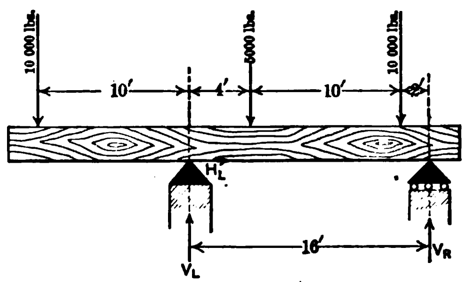


FIG. 20.

gives the value of V_R and completes the solution of the problem. The equation follows:

$$-10{,}000+21{,}250-5000-10{,}000+V_R=0; \quad \therefore\ V_R=+3750 \text{ lbs.},$$

and acts upward as shown.

To check this value apply $\Sigma M=0$, using the left point of support for the origin of moments. The expression thus obtained is

$$-10{,}000\times 10+5000\times 4+10{,}000\times 14-16V_R=0; \quad \therefore\ V_R=+3750 \text{ lbs.},$$

which checks the value obtained by the application of $\Sigma V=0$ and hence checks the value of V_L since this was used in the original determination of V_R.

Problem. Compute horizontal and vertical components of reactions on beam, shown in Fig. 21, neglecting weight of beam.

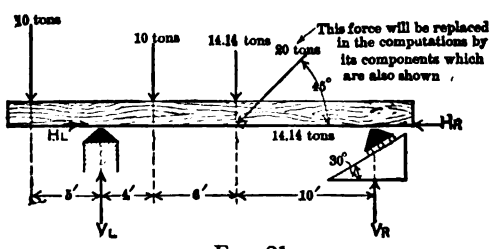


FIG. 21.

Solution. In this problem V_L and H_L are independent of each other in magnitude and direction and each may be assumed as acting in either direction. V_R and H_R are, however, mutually related both in

direction and magnitude since their resultant must act at right angles to the supporting surface, and hence make an angle of 60° with the horizontal. To fulfil this condition if V_R is assumed as upward, H_R must be assumed to the left. The ratio of their magnitude equals the ratio of the sides of a 30° triangle, as indicated by Fig. 22, hence $V_R = H_R \cot 30° = 1.73H_R$.

To solve this problem apply the equation $\Sigma M = 0$, taking moments about the point of application of the right-hand reaction. The following equation results:

$$-10\times25+20V_L-10\times16-14.14\times10=0.$$

The solution of this gives $V_L = +27.57$ tons; $\therefore$ V_L acts upward as assumed.

It will be noticed that the lever arms about the origin of moments of all the horizontal forces are zero, hence these terms do not appear in the equation. Had the inclined force been resolved at the *top* instead of the *bottom* of the beam, this condition would not have existed, but the value of the reaction would not have been changed since the moment of the horizontal component would have been neutralized by the change in the moment of the vertical component due to its altered lever arm.[1]

The equations $\Sigma V = 0$ may now be used. This gives the following expression,

$$-10+27.57-10-14.14+V_R=0,$$

hence $V_R = +6.57$ tons and acts as shown.

From Fig. 22 it is evident that $H_R = V_R \tan 30° = 0.577V_R$; $\therefore$ $H_R = 0.577\times 6.57 = 3.79$ tons.

The application of $\Sigma H = 0$ completes the solution by giving the value of H_L. The equation is $H_L - 14.14 - 3.79 = 0$; hence $H_L = 17.93$ tons and acts to the right.

To check the value of V_R take moments about the left point of support. This gives the following expression:

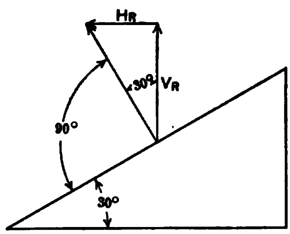


FIG. 22

$$-10\times5+10\times4+14.14\times10-20V_R=0,$$

[1] The device of resolving a force into its components at a point where the lever arm of one of the components is zero is a very useful one, and frequently saves considerable labor. Its correctness is evident since the effect of a force upon a body as a whole always equals that of its components no matter at what point the force is resolved, nor what may be the direction or length of the lever arms of the components, hence if the lever arm of one of the components is zero the moment of the force equals the moment of the other component.

whence $V_R = +6.57$ tons, thus checking the value previously obtained, and in consequence the value of V_L.

As an independent check of H_R and H_L cannot readily be made a second computation of their value should be carried through, or the original computations carefully reviewed, the former being the safest method.

Problem. Compute horizontal and vertical components of the reactions for the truss shown in Fig. 23 for an assumed wind pressure of 30 lbs. per square foot on a vertical surface.

Solution. Since the slope of the roof surface in this problem is about 30°, it will be assumed that the normal intensity of the wind pressure is 20 lbs. per square foot. (See table in Art. 13, Hutton's formula.) The roof trusses are 20 ft. between centres, hence the portion of the area of the windward side of the building supported by one truss has a length of

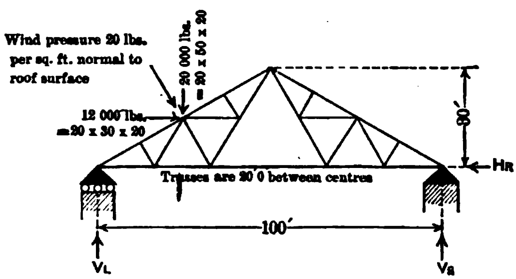


FIG. 23.

20 ft. for intermediate trusses, and 10 ft. for end trusses. The reactions upon an intermediate truss will be computed.

Using the method of Art. 13 the horizontal and vertical components of the total wind pressure on the windward side are found to be as follows:

P_h = intensity of normal pressure multiplied by the vertical projection of roof surface $= 20 \times 30 \times 20 = 12{,}000$ lbs.

P_v = intensity of normal pressure multiplied by the horizontal projection of surface $= 20 \times 50 \times 20 = 20{,}000$ lbs.

The truss may now be considered as loaded with the two forces of 20,000 lbs. and 12,000 lbs. acting at centre of windward surface, and the reactions due to these forces computed in the following way:

Applying $\Sigma M = 0$ about right end gives $100V_L + 12{,}000 \times 15 - 20{,}000 \times 75$, whence $V_L = +13{,}200$ lbs., acting up as assumed.

Applying $\Sigma H = 0$ gives $12{,}000 - H_R = 0$, whence $H_R = +12{,}000$ lbs., acting to left as assumed.

Applying $\Sigma V = 0$ gives $13{,}200 - 20{,}000 + V_R = 0$, hence $V_R = +6800$ lbs., acting up as assumed.

Applying $\Sigma M = 0$ about left end as a check gives $-100V_R + 20{,}000$

$\times 25+12{,}000\times 15=0$, whence $V_R=+6800$ lbs., acting up as assumed and agreeing with value previously obtained.

Problem. Compute horizontal and vertical components of the reactions on crane shown in Fig. 24. Neglect weight of structure itself.

Solution. The direction of the reaction at the top of the crane is fixed by the tie-rod, hence V_L and H_L cannot be assumed to act at random but must be so chosen that their resultant will act along the tie-rod. Their magnitude will, of course, be equal since the tie-rod makes an angle of 45° with the horizontal.

Applying $\Sigma M=0$ about the bottom gives $-35H_L+5000\times(20+30)+20{,}000\times 40=0$, hence $H_L=+30{,}000$ lbs., acting as assumed. Since the two components of the tie-rod stress are equal $V_L=30{,}000$ lbs., also acting as assumed.

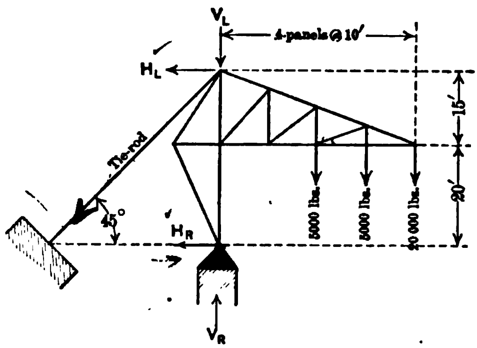


FIG. 24.

Applying $\Sigma H=0$, using the value previously found for H_L, gives $30{,}000+H_R=0$, hence $H_R=-30{,}000$ lbs. acting to the *right* and *not* as assumed.

Applying $\Sigma V=0$, using the value previously found for V_L, gives $-30{,}000-5000-5000-20{,}000+V_R=0$, hence $V_R=+60{,}000$ lbs. acting up as assumed.

Applying $\Sigma M=0$, about the top as a check gives $+35H_R+5000\times 20+5000\times 30+20{,}000\times 40=0$, hence $H_R=-30{,}000$ lbs. checking the value previously obtained.

It is always advisable to assume the reactions as acting in their probable directions to avoid complications. The opposite assumption was made for H_R in above problem in order to illustrate the solution with an incorrect assumption. The results will be found to agree in any case, provided the work is correctly done, but it is confusing to have the reaction incorrectly indicated on the sketch. Sometimes, however, it is impossible to foretell the actual direction of a reaction.

In this problem the actual value of the reaction at the top should be found, since this gives the tension in the tie-rod.

This value $= \frac{30{,}000}{\sin 45^\circ} = 42{,}430$ lbs. approximately.

This should equal $\sqrt{V_L^2 + H_L^2}$, which may be used as a check.

26. Shear and Bending Moment Defined. *Shearing force or shear* at any section of a body is that force which tends to produce slipping along the given section.

The bending moment at any section of a body due to a set of co-planar forces is the resultant moment about an axis passing through the centre of gravity of the section, of all the forces on either side of the section, it being understood that the section and the axis are perpendicular to the plane of the forces.

Fractures due to shear are due either to transverse fracture of the grains or fibres, or to the slipping of the fibres upon each other. Of the ordinary structural materials wood is the only one of a fibrous character and shearing failures in this material ordinarily occur by longitudinal slipping of the fibres.

Fractures due to bending are caused by longitudinal failure of the fibres, either by tension or crushing.

27. Method of Computation, Shear and Bending Moment. The magnitude of the shear upon a given section due to a set of co-planar forces may be readily computed as follows: Resolve each force into two components parallel and perpendicular, respectively, to the given section. The algebraic sum of the components parallel to the section of all the forces upon either side of the section equals the shear. That either side of the section may be considered is evident from the fact that for structures in equilibrium $\Sigma Y = 0$, hence the algebraic sum of the forces on one side of the section and parallel to the Y-axis must be equal in magnitude and opposite in direction to the corresponding term for the other side of the section.

The magnitude of the bending moment upon a given section due to a set of co-planar forces may be computed by resolving the forces into horizontal and vertical components. For this case, however, it is necessary to include the moments of both sets of components, though again it is immaterial which side of the section is considered in computing the moment.

28. Signs of Shear and Bending Moment. The signs for shears and bending moments must be used with care or errors will occur. Any reasonable convention may be adopted, but it

should be carefully observed that positive shear may represent forces acting in exactly opposite directions and that positive bending moment may represent either clockwise or counter-clockwise moment, depending in both cases upon the side of the section considered in making the computation. The distinction between the moment of forces in general as used, for example, in determining reactions, and the moment upon a cross-section of a beam should be carefully observed. In the former case clockwise moments should always be taken as of the same sign, since the effect of such moments upon the body as a whole is the same no matter upon what part of the body they may act. In a beam, however, clockwise moment upon the left of a given section produces the same effect upon the fibres as does counter-clockwise moment upon the right. In both cases compression is produced in the fibres of the upper portion of the section and tension in those of the lower portion.

29. Shear and Moment, Common Cases. In ordinary practice it is seldom necessary to compute shears or moments, except for vertical sections of horizontal beams and trusses carrying *vertical* loads. For such cases the following conventions may be adopted.

Shear. The shear upon a vertical section of a beam or truss equals the algebraic sum of all the outer forces (including reactions) upon either side of the section. It is positive when the resultant is *upward* on the *left* of the section or *downward* on the *right.*

Moment. The moment upon a vertical section of a beam or truss equals the algebraic sum of the moments of all the outer forces (including reactions) upon either side of the section, about the neutral axis of that section. It is positive when the moment of the forces on the left of the section is *clockwise*, or when the moment of the forces on the *right* of the section is *counter-clockwise.*

30. Curves of Shear and Moment Defined and Illustrated. A curve of shears or of moments is a curve the ordinate to which at any section represents the shear or moment at that section due to the applied loads. If the load be uniformly distributed the curve may be a continuous smooth curve, a series of smooth curves, or a series of straight lines. If the loading consists of a series of concentrated loads the curve will always be com-

posed of a series of straight lines. If the loading be a combination of concentrated and distributed loads the curves may be composed of a combination of straight and curved lines.

It should be thoroughly understood that these curves represent the effect of loads which are *fixed* in *magnitude* and *position*. The shear and moment due to a set of moving loads constantly

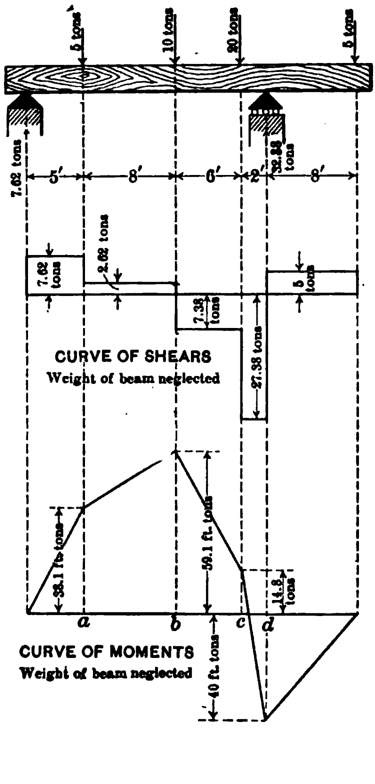
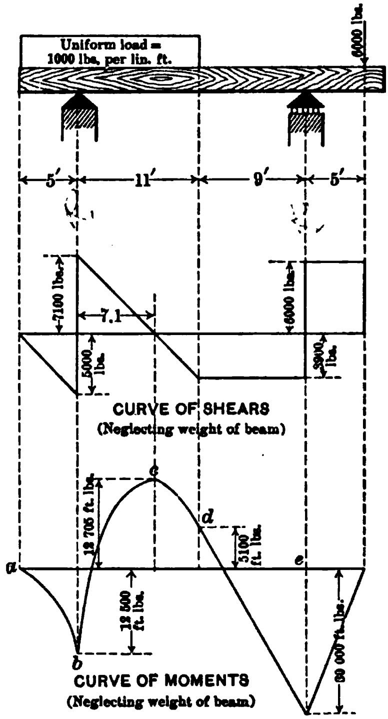


FIG. 25.



Curve $a\ b$ is a parabola
Equation is $y = -500x^2$
Curve $b\ c\ d$ is a parabola
Equation is $y = -500x^2 + 12\ 100\ (x-5)$

FIG. 26.
(Read Art. 31 before studying this figure).

vary and hence cannot be represented by such curves except for a certain definite position of the loads. The effect of moving loads is shown more clearly by influence lines which are explained later. Typical curves of shears and moments are shown in the figures which follow.

It should be noticed that in all cases the ordinate to the curve

of shears at any section equals the algebraic sum of the forces acting on either side of the section, and that the curve of moments reaches its maximum positive and maximum negative values at points where the curve of shear crosses the axis.

This latter relation always exists and is demonstrated in Art. 33.

The computation of the values of the ordinates to the curve of moments at points *a*, *b*, *c*, and *d* of Fig. 25 are given below for illustration.

At a,	7.62×5	$= +38.10$	ft.-tons.
b,	$7.62 \times 13 - 5 \times 8$	$= +59.06$	"
c,	$7.62 \times 19 - 5 \times 14 - 10 \times 6$	$= +14.78$	"
d,	-5×8	$= -40.0$	"

Note that a point of maximum or minimum moment occurs in all cases where the curve of shears crosses the axis and that the moment curve is a straight line over any portion of beam where shear is constant.

31. Shear and Moment. Distributed Load. In determining reactions it has been stated that a distributed load may be replaced by its resultant and the latter used as a concentrated load. This method is *incorrect* for shear and moment, and should never be used for such cases unless the distributed load lies wholly on *one side* of the section under consideration. The reason for this may readily be seen. Both shear and moment are functions of the forces on *one side* only of a section, and all such forces must be included in the determination of either of these quantities. It is evident that if the structure be loaded with a distributed load its resultant may act on either side of a given section, say on the right, while a considerable portion of the actual load may be on the left. If the shear or moment be computed for the forces on the left of the section with the distributed load replaced by its resultant, the serious error of neglecting a considerable portion of the loads will be made. For reactions, on the other hand, it is the influence of the load as a whole which is to be considered, hence the resultant may properly be used. To illustrate the difference between the correct curves of shear and moment for the case of a beam carrying a uniformly distributed load, and the same curves if drawn in accordance with the erroneous assumption that the load may be replaced in magnitude and position by a concentrated load, see Fig. 27.

32. Shear and Moment. Uniformly Varying Load. It is frequently necessary to determine shears and moments for a beam or girder loaded with a uniformly varying load. Such a condition may occur with a vertical member subjected to

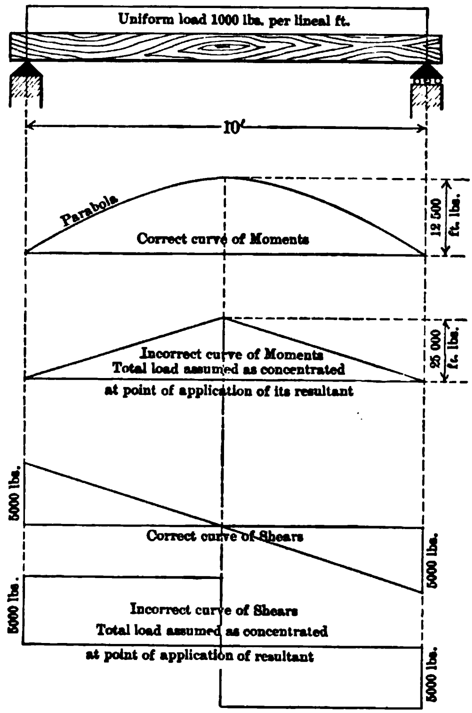
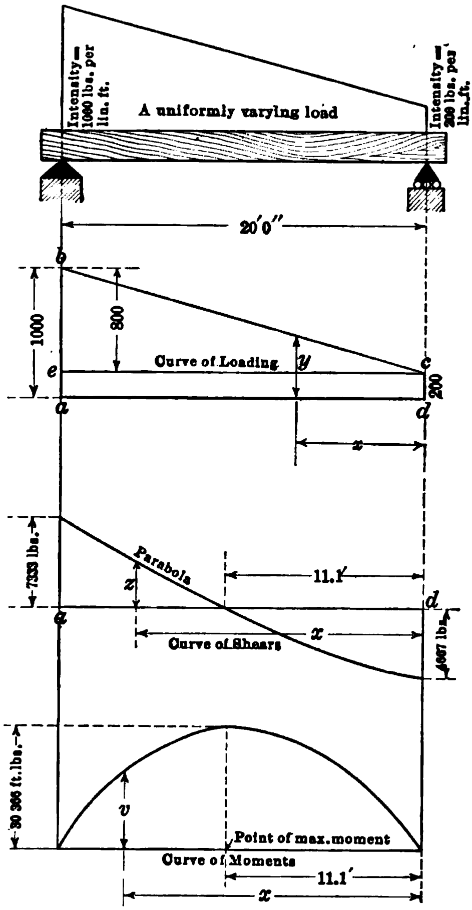


FIG. 27.

hydrostatic pressure, as in a canal lock, or in a diagonal floor girder in a building.

The curves of shear and moment for such a girder are shown in Fig. 28, and the necessary computations follow.

Let the load be represented in intensity by the trapezoid *abcd*, the area of which represents the total load on the beam. If the trapezoid be divided into two parts by a line *ce* parallel to the axis *ad* the effect of each portion may be treated separately and the problem simplified.



$$y = 200 + \frac{800}{20}x$$

Note that the curve of shears corresponds to integral of the loading curve, hence

$$z = \int y\,dx = \int (200+40x)dx$$
$$= 200x + 20x^2 + C_1$$
$$-C_1 = 4667 = \text{value of } z \text{ when } x = o$$
$$\therefore z = 200x + 20x^2 - 4667$$

Note that the curve of moments corresponds numerically to integral of curve of shears, hence

$$v = \int z\,dx = 4667x - 100x^2 - \frac{20}{3}x^3$$

FIG. 28.

Magnitude of force represented by triangle bce =
$\frac{800}{2} \times 20$ = ... 8000 lbs.
Magnitude of force represented by rectangle $adce$ =
200×20 = ... 4,000 "

Total load = ... 12,000 lbs.
= area of trapezoid $abcd$.

Reaction. The computation of the reactions should be divided into two operations: the determination of the reactions due to the load represented by the rectangle *aecd* and the determination of the reactions due to the load represented by the triangle *bec*.

For the first case both reactions $=\frac{200\times20}{2}=2000$ lbs. $=V$.

To determine the reactions due to the load represented by triangle *bce* it is advisable to determine the position of the resultant of this load. This passes through the centre of gravity of the triangle and hence is $\frac{20}{3}$ ft. from the line *ab* and $\frac{40}{3}$ ft. from *cd*. The left reaction V_L'' due to this load may now be determined by applying $\Sigma M=0$ about right hand end of beam. The following expression results: $-8000\times\frac{40}{3}+V_L''\times20=0$, hence $V_L''=5333$ lbs. The total left reaction, V_L, therefore equals $V_L''+V=7333$ lbs.

To obtain the right reaction apply $\Sigma V=0$. This gives $7333-12{,}000+V_R=0$ hence $V_R=4667$ lbs., which may be checked by applying $\Sigma M=0$ about the left end of the beam.

The curve of shears may now be drawn. Its equation referred to rectangular axes passing through point *d* with x positive to the left and z positive upwards is $z=-4667+200x+20x^2$, in which the term $200x$ equals the area of a rectangle of height *cd* and length x, and the term $20x^2$ equals the area of that portion of the triangle, *bce*, comprehended between its vertex, *c*, and a vertical line drawn at a distance x from the vertex. This curve cuts the axis at a point 11.1 ft. from right end, as may be seen by placing $z=0$ and solving for x.

The curve of moments may be obtained in a similar manner. Its equation referred to the same origin is $v=4667x-\frac{200x^2}{2}-\frac{20x^3}{3}$. This equation may be written directly from the shear equation by multiplying each term in the latter, which represent forces, by the distance of the particular force from the section. Thus, 4667 equals the right reaction and hence should be multiplied by x; $200x$ equals that portion of the load represented by a rectangle extending a distance, x, from the right reaction, and hence should be multiplied by $\frac{x}{2}$; $20x^2$ equals that portion of the

load represented by a triangle of length x, and with its vertex at the right reaction, and hence should be multiplied by $\frac{x}{3}$.

33. Location of Section of Maximum Moment. It is a well-established principle of mechanics that the first derivative of the moment equals the shear, hence the moment must have either a minimum or a maximum value at every section where the curve of shears crosses the axis of the beam. The following rule may therefore be stated: The maximum moment always occurs at a section where the curve of shears crosses the axis of the beam; i.e., where the shear equals zero.

This rule may also be proven by the use of the theorem of Art. 34, since it is evident that the moment M_a begins to diminish when S_a changes from positive to negative, i.e., passes through zero.

The reader will observe that if the equation for the curve of moments in Art. 32 be differentiated with respect to x the equation for the curve of shears will be obtained. In the light of what has just been stated, this is correct, and such a result should always be found.

The converse of this is also true, viz.:

That the moment curve is the integral of the shear curve with respect to x. It follows that the ordinate to the curve of moments at any section equals the area of the shear curve between the end of the beam and the section. An inspection of the numerous shear and moment diagrams on the following pages will show that this relation occurs in every case. The student in testing this by integration must not forget the constant of integration.

34. Theorem for Computing Moments. In computing moments at a number of consecutive points, as is often necessary in dealing with concentrated loads, the following theorem may be used to great advantage:

The moment at any section b of a structure loaded with parallel forces, either concentrated or distributed, is equal to the moment at any other section a, at a distance x from b, plus (algebraically) the shear at a multiplied by x, plus (algebraically) the moment about b of the loads between a and b. This may be expressed as follows:

Let S_a = shear at section a.
M_a = moment at section a.
M_b = moment at section b.
x = distance between section a and section b measured at right angles to the line of action of the forces.
M_x = moment about b of forces between a and b.

Then $M_b = M_a \pm S_a x \pm M_x$.

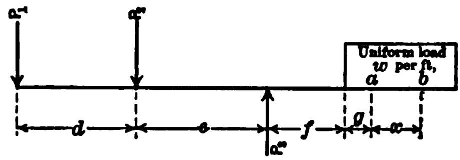


FIG. 29.

This may be proven in the following manner:

$$M_a = -P_1(d+e+f+g) - P_2(e+f+g) + P_3(f+g) - \frac{wg^2}{2},$$

$$M_b = -P_1(d+e+f+g+x) - P_2(e+f+g+x) + P_3(f+g+x) - w\frac{(g+x)^2}{2}.$$

$\therefore$ by subtraction;

$$M_b - M_a = -P_1 x - P_2 x + P_3 x - \frac{w}{2}(2gx + x^2).$$

But $\qquad P_3 - P_1 - P_2 - wg = S_a \quad \text{and} \quad \frac{wx^2}{2} = M_x.$

$$\therefore \quad M_b = M_a + S_a x - M_x. \qquad (9)$$

This solution is perfectly general since no restrictions were imposed upon character or position of the loads.

35. Beams Fixed at Ends. The beams hitherto dealt with have been supported at two points and have been statically determined. Sometimes, however, beams are used which are fixed at both ends by being built into the masonry or otherwise and are statically undetermined. Complete treatment of such beams may be found in standard books on mechanics and will not be repeated here, although attention should be called to the fact that such beams are much stronger than beams of the same size which are merely supported at the ends.

A beam fixed at one end is also indeterminate with respect to the reactions, but the moment and shear at any section of the projecting end can be computed without difficulty.

Such a beam is shown in Fig. 30, in which an assumed distribution of the reactions is indicated, viz., a uniformly varying downward reaction, the resultant of which is R_2, and another uniformly varying upward reaction the resultant of which equals R_1. It is evident that $R_1 = R_2 + P_1 + P_2 + W$, and that the moment of R_2 about the point of application of R_1 must equal the moment about the same point of P_1, P_2, and W. The actual distribution of the reaction depends upon the

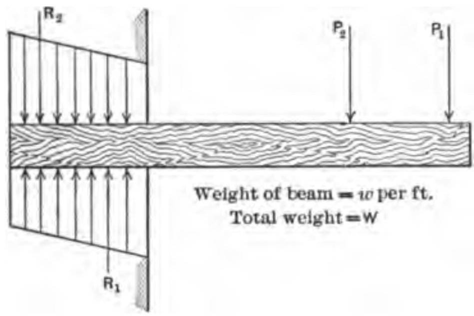


FIG. 30.

relative elasticity of beam and masonry and will not be discussed. The maximum bending moment and shear occur at, or very near, the edge of the masonry and can be computed with no greater error than for ordinary beams resting on masonry abutments, hence a beam of this sort can be designed with comparative certainty, provided reasonable provision is made at the ends for carrying the reactions.

36. Effect of Floor Beams. Reactions, moments, and shears upon a structure *as a whole* are uninfluenced by the internal construction. For example, the reactions at the ends of a structure due to a given loading are the same whether it is a simple beam or is made up of trusses, floor beams, and stringers. This immunity, however, does not extend to the individual members of the structure which are influenced to a marked degree by the construction adopted. In the case of an ordinary bridge composed of trusses or girders, floor beams, and stringers, the shears and moments on the trusses vary considerably from those which

would exist if there were no floor beams, and this applies also to the reactions if floor beams be not used at the ends.

The effect of floor beams is to load the main girders or trusses with loads at definite points. This is clearly shown by the figures accompanying Art. 1. The load reaches the stringers through the floor, is carried by them to the floor beams, and thence goes to the main girders. In consequence the girders carry only concentrated loads except for their own weight, and the curves of shear and moment for the applied loads are composed of straight lines.

37. Typical Curves of Shear and Moment. A few curves of shear and moment have already been drawn to illustrate the

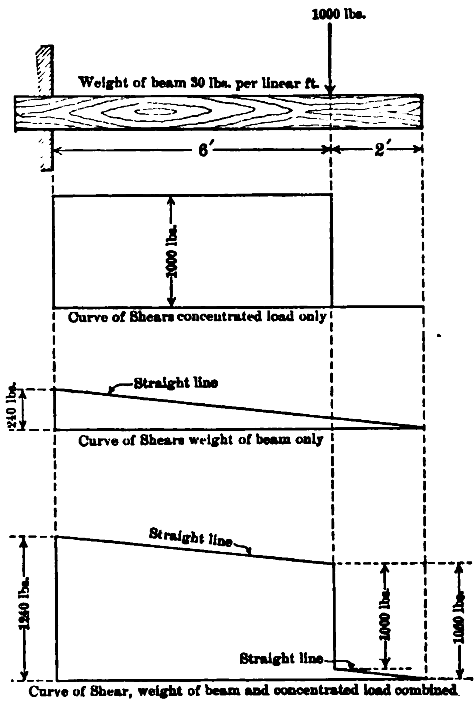


FIG. 31.

text. In the figures which follow, the attempt has been made to cover a wide range of cases. The beginner should draw curves for similar cases, changing the data to avoid copying, until he understands the subject thoroughly.

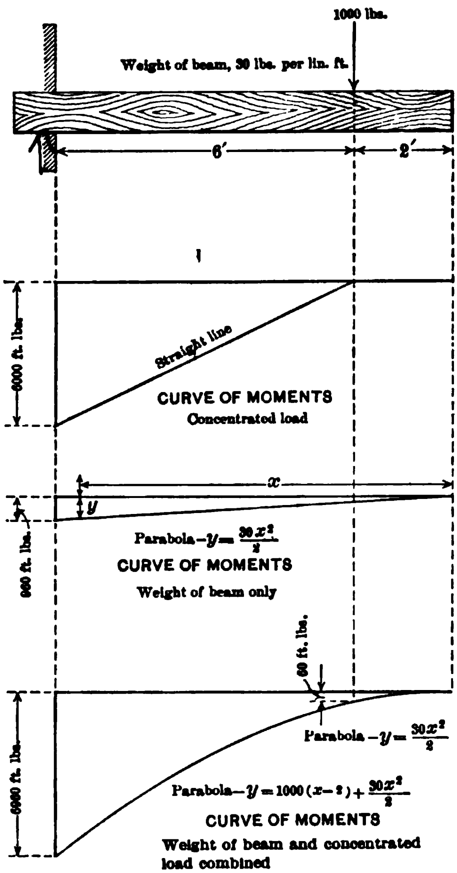

1000 lbs.
Weight of beam, 30 lbs. per lin. ft.
6′
2′
6000 ft. lbs.
Straight line
CURVE OF MOMENTS
Concentrated load
x
y
960 ft. lbs.
Parabola–$y = \frac{30x^2}{2}$
CURVE OF MOMENTS
Weight of beam only
60 ft. lbs.
Parabola –$y = \frac{30x^2}{2}$
6960 ft. lbs.
Parabola–$y = 1000(x-2) + \frac{30x^2}{2}$
CURVE OF MOMENTS
Weight of beam and concentrated load combined

Fig. 32.

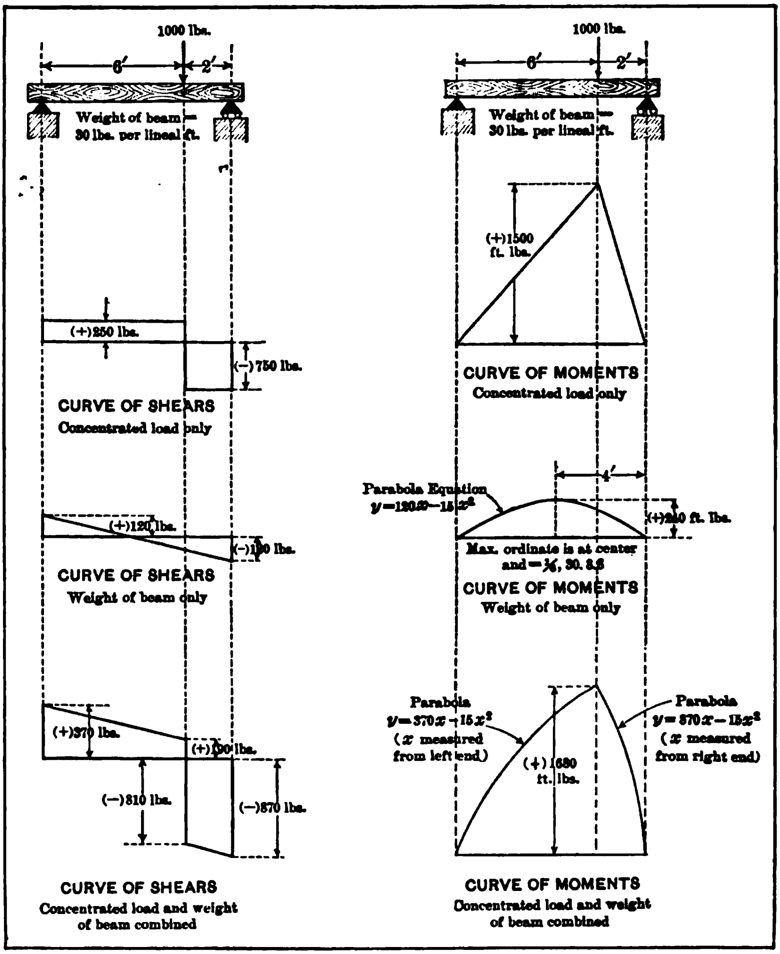

1000 lbs.
6'
2'
Weight of beam = 30 lbs. per lineal ft.
(+)250 lbs.
(−)750 lbs.
CURVE OF SHEARS
Concentrated load only
(+)120 lbs.
(−)120 lbs.
CURVE OF SHEARS
Weight of beam only
(+)370 lbs.
(+)190 lbs.
(−)810 lbs.
(−)870 lbs.
CURVE OF SHEARS
Concentrated load and weight of beam combined
1000 lbs.
6'
2'
Weight of beam = 30 lbs. per lineal ft.
(+)1500 ft. lbs.
CURVE OF MOMENTS
Concentrated load only
4'
Parabola Equation
y=120x−15x²
(+)240 ft. lbs.
Max. ordinate is at center and = ⅛, 30. 8.8
CURVE OF MOMENTS
Weight of beam only
Parabola
y=370x−15x²
(x measured from left end)
Parabola
y=870x−15x²
(x measured from right end)
(+)1680 ft. lbs.
CURVE OF MOMENTS
Concentrated load and weight of beam combined

Fig. 33.

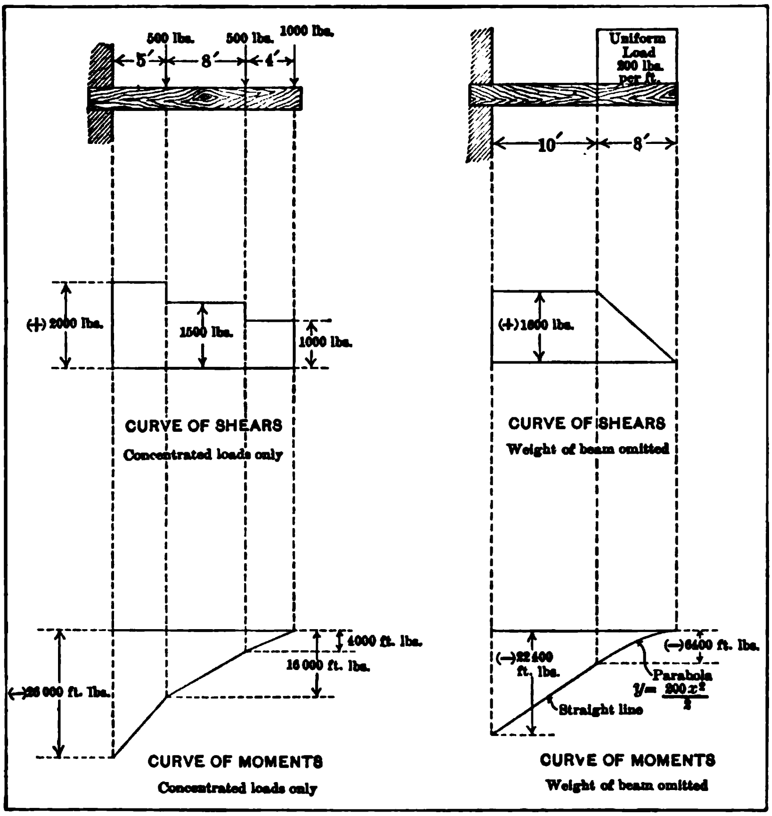

500 lbs.
500 lbs.
1000 lbs.
5′
8′
4′
(+) 2000 lbs.
1500 lbs.
1000 lbs.
CURVE OF SHEARS
Concentrated loads only
4000 ft. lbs.
16 000 ft. lbs.
(−) 26 000 ft. lbs.
CURVE OF MOMENTS
Concentrated loads only
Uniform Load 200 lbs. per ft.
10′
8′
(+) 1600 lbs.
CURVE OF SHEARS
Weight of beam omitted
(−) 22 400 ft. lbs.
(−) 6400 ft. lbs.
Parabola
$y = \frac{200x^2}{2}$
Straight line
CURVE OF MOMENTS
Weight of beam omitted

Fig. 34.

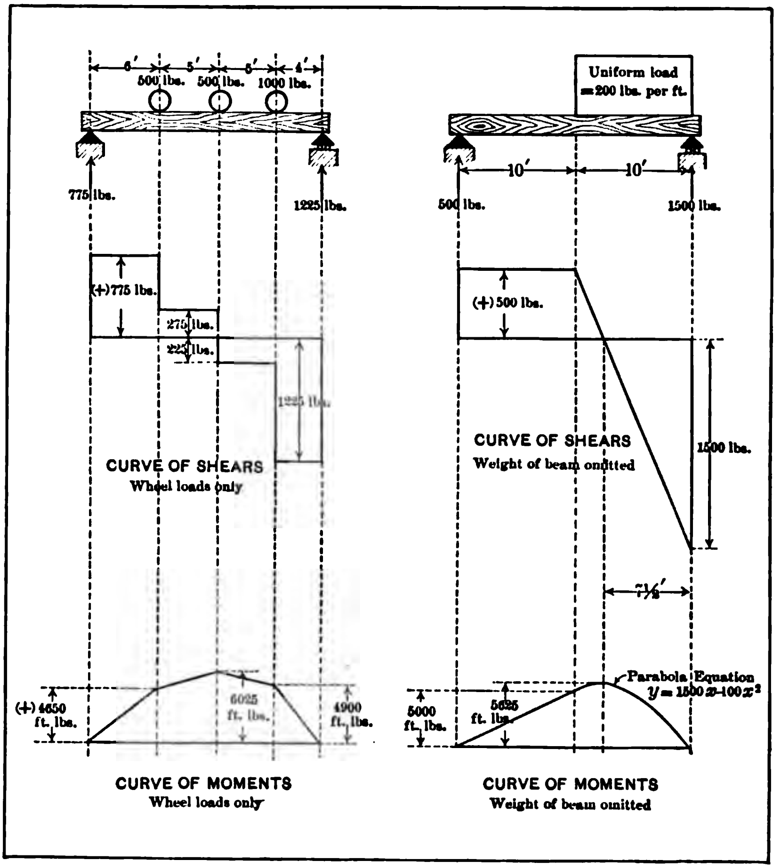

6'
5'
5'
4'
500 lbs.
500 lbs.
1000 lbs.
775 lbs.
1225 lbs.
(+) 775 lbs.
275 lbs.
225 lbs.
1225 lbs.
CURVE OF SHEARS
Wheel loads only
(+) 4650 ft. lbs.
6025 ft. lbs.
4900 ft. lbs.
CURVE OF MOMENTS
Wheel loads only
Uniform load = 200 lbs. per ft.
10'
10'
500 lbs.
1500 lbs.
(+) 500 lbs.
CURVE OF SHEARS
Weight of beam omitted
1500 lbs.
7½'
Parabola Equation
y = 1500 x-100 x²
5000 ft. lbs.
5625 ft. lbs.
CURVE OF MOMENTS
Weight of beam omitted

FIG. 35.

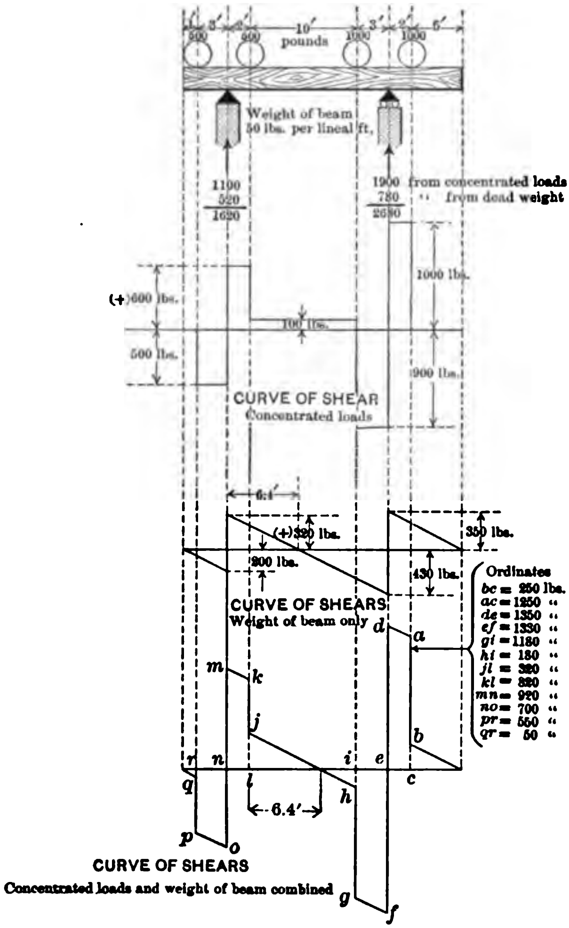
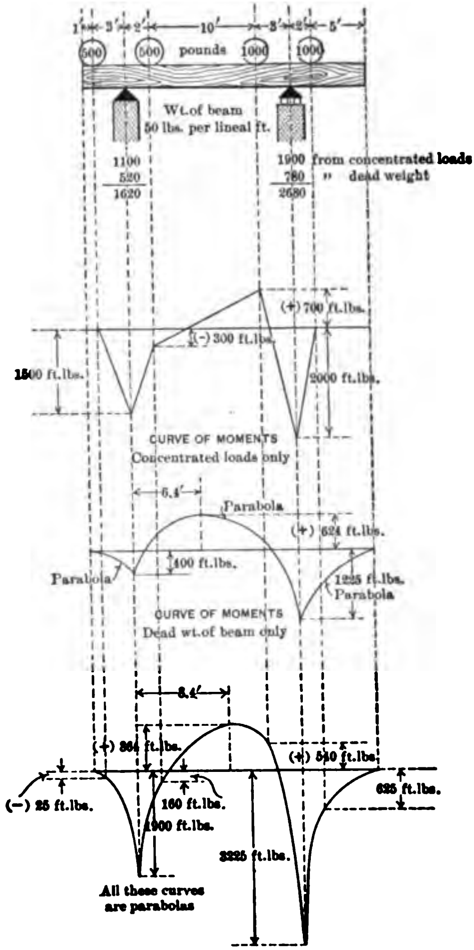

1′ 3′ 2′ 10′ 3′ 2′ 5′
500 900 1000 1000
pounds
Weight of beam
50 lbs. per lineal ft.
1100
520
1620
1900 from concentrated loads
780 " from dead weight
2680
1000 lbs.
(+)600 lbs.
100 lbs.
500 lbs.
900 lbs.
CURVE OF SHEAR
Concentrated loads
6.4′
(+)320 lbs.
350 lbs.
200 lbs.
430 lbs.
CURVE OF SHEARS
Weight of beam only
Ordinates
bc = 250 lbs.
ac = 1250 "
de = 1350 "
ef = 1330 "
gi = 1180 "
hi = 180 "
jl = 320 "
kl = 820 "
mn = 920 "
no = 700 "
pr = 550 "
qr = 50 "
a b c d e f g h i j k l m n o p q r
6.4′
CURVE OF SHEARS
Concentrated loads and weight of beam combined

FIG. 36.



CURVE OF MOMENTS
Concentrated loads and weight of beam combined.

Fig. 37.

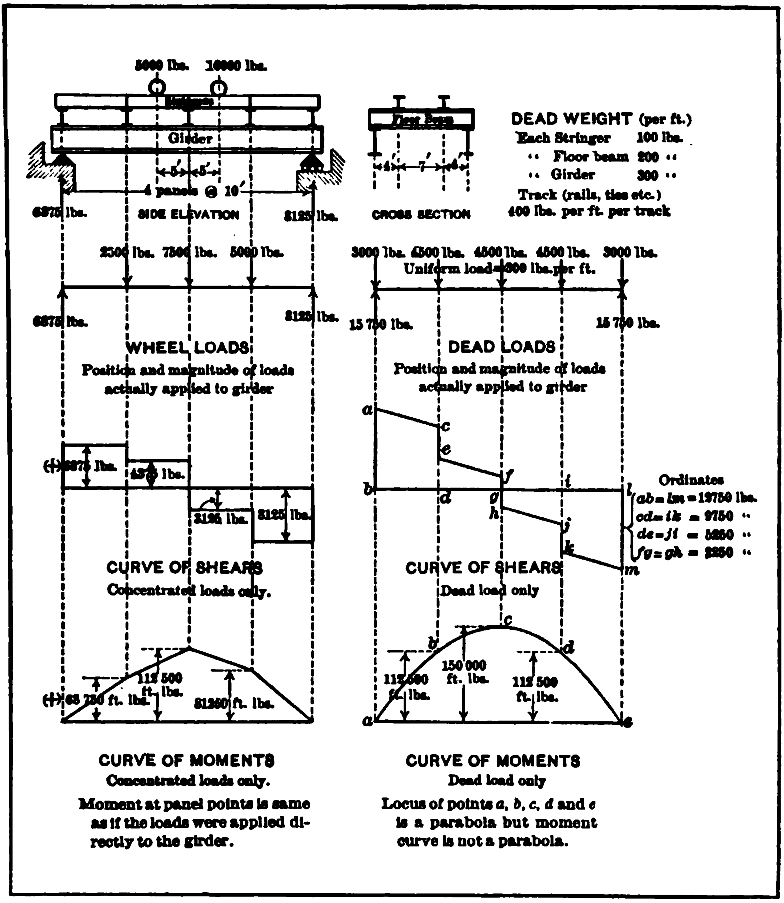


FIG. 38.

Note.—Floor beams are ordinarily riveted to sides of girders. Above construction is adopted here for sake of clearness.

38. Influence Lines and Tables Defined. In the determination of maximum shears, moments, reactions, and other functions due to moving loads, it is frequently useful to study the effect of a load of unity as it moves along the structure. This may be done graphically by plotting a line called an influence line, or analytically by preparing an influence table in which are set

down the values of the function under consideration when the load is at various governing points, such as the panel points of a truss bridge. The following simple illustration shows clearly the character of line and table.

Distance of Load from Right Reaction.	Shear at a.
1 ft.	+1/6
2 ft.	+2/6
3 ft.	+3/6
3.9 ft.	+39/60
4.1 ft.	−19/60
5 ft.	−1/6

Influence table for shear at a of beam shown in Fig. 39.

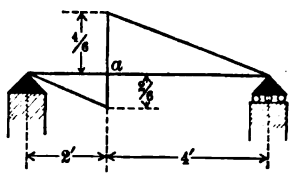


FIG. 39.—Influence line for simple beam. Shear at a.

The influence line in this case is the locus of the values in the second column of the influence table and is merely the graphical representation of the equation for the shear at a due to a load of unity passing along the beam. If x be the distance of the load from the left reaction and y the ordinate, the equations of the influence line will be as follows:

$$y = -\frac{x}{6}, \ x \text{ varying between 0 and } 2'$$

and

$$y = \frac{6-x}{6}, \ x \text{ varying between } 2' \text{ and } 6'.$$

The difference between an influence line and the curves given in the preceding articles should be carefully observed. A curve of shears, or moments, is a curve, the ordinate to which at any point shows the shear, or moment, at that point caused by a set of loads, fixed in magnitude and position. The ordinate to the influence line shows instead the shear or moment at the section for which the influence line is drawn, due to a load of unity acting at the point where the ordinate is measured. The examples in Art. 39 serve to illustrate influence lines for the more common cases of simple beams and girders.

The actual employment of influence lines and tables in practice seldom occurs except for complicated structures where they are frequently almost indispensable. In this book the influence line will, however, be used with freedom, partly for purposes of illustration and demonstration and partly that the student may better familiarize himself with the behavior of various structures under moving loads.

39. Examples of Influence Lines. *a. Simple Beams and Girders.*

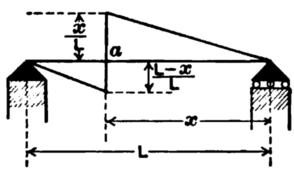


FIG. 40.—Influence line for shear at section *a*.

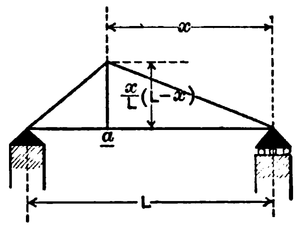


FIG. 41.—Influence line for moment at section *a*.

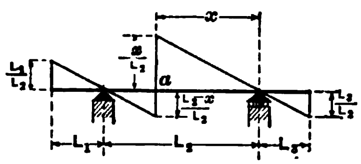


FIG. 42.—Influence line for shear at section *a*.

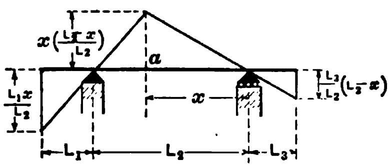


FIG. 43.—Influence line for moment at section *a*.

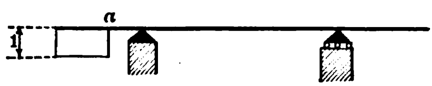


FIG. 44.—Influence line for shear at section *a*.

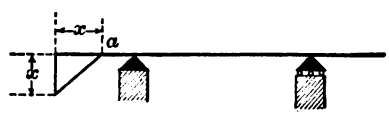


FIG. 45.—Influence line for moment at section *a*.

b. Girders with Loads Applied through Floor Beams, as in Fig. 46. Note that the usual form of construction for such bridges

is that in which the floor beams are riveted to the girder webs and the stringers to the floor beam webs. The type shown in

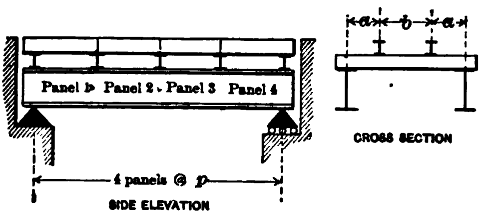


FIG. 46.

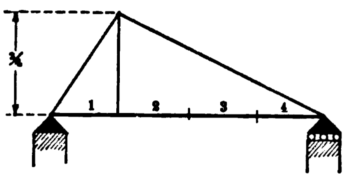


FIG. 47.—Influence line for shear in panel 1.

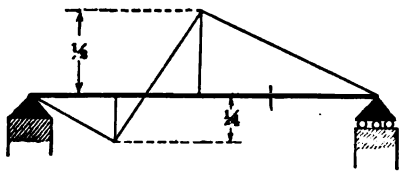


FIG. 48.—Influence line for shear in panel 2.

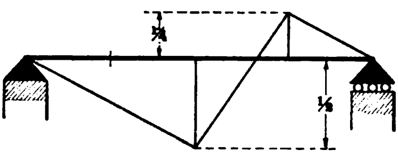


FIG. 49.—Influence line for shear in panel 3.

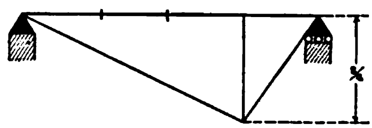


FIG. 50.—Influence line for shear in panel 4.

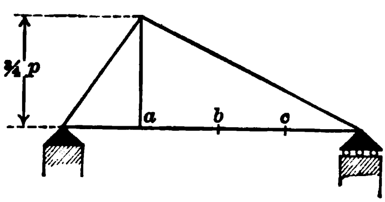


FIG. 51.—Influence line for moment at panel point *a*.

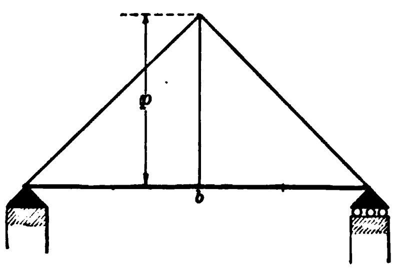


FIG. 52.—Influence line for moment at panel point *b*.

the figure is chosen here for clearness in presentation. The influence lines, moments, shears, etc., would be identical in the two cases.

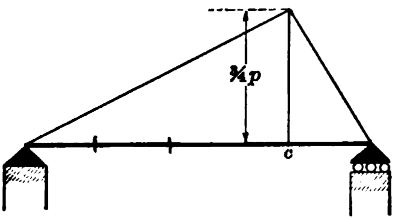


FIG. 53.—Influence line for moment at panel point c.

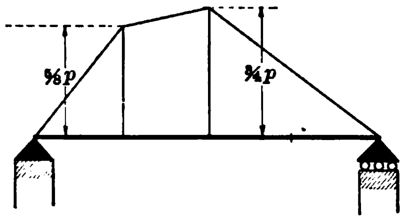


FIG. 54.—Influence line for moment at center of panel 2.

40. Properties of the Influence Line. The following theorems may often be used to advantage.

1. The value of a given function due to a single load in a fixed position equals the product of the magnitude of the load and the ordinate to the influence line measured at the point where the load is placed. This needs no proof, but follows directly from the definition of the influence line.

2. The value of a given function due to a uniformly distributed load equals the product of the *intensity* of the load and the *area* bounded by the axis of the beam, the influence line and the ordinates drawn through the limits of the load. If this area be partially positive and partially negative the algebraic sum of the two should be used.

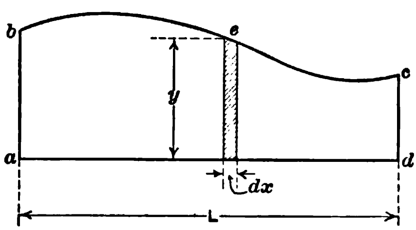


FIG. 55.

This theorem may be proven as follows:

Let bec represent an influence line for a portion of a given structure of length L. Let w equal the intensity of a uniformly distributed load.

Then the total load on a section of length $dx = wdx$ and the effect of this portion of the load upon the given function $= wydx$. Integrating between the limits 0 and L gives

$$w\int_0^L ydx$$

for the effect of a load covering the entire distance L. But ydx is the area of the infinitesimal strip subtended by dx, and $\int_0^L ydx$ is the area *abecd*, hence $w\int_0^L ydx = w \times$ area *abecd*.

3. The value of a given function due to a set of concentrated loads equals the algebraic sum of the product of each load and its corresponding ordinate to the influence line. This is a corollary of 1.

41. Neutral Point. The influence lines shown in Figs. 47 to 50 inclusive cross the axis of the beam in each case except for shear in the end panels. The point of intersection is called the neutral point since a single load placed at this point produces no shear in the panel where the intersection occurs.

The neutral point for the end panels is at the ends of the beams.

42. Position of Loads for Maximum Shear and Moment at a Definite Section. The following important laws may be deduced from the influence lines given in Art. 39.

1. For a simple beam supported at the ends a single concentrated load causes maximum shear at a section when placed an infinitesimal distance on one or the other side of the section, and maximum moment when placed at the section. A uniformly distributed live load produces maximum shear at a section when applied over the entire distance between the section and one or the other end of the beam, and maximum moment when applied over the entire length of the beam,

2. In an end-supported girder or truss loaded by means of floor beams, a single concentrated load produces maximum shear in a panel when placed at the end of the panel adjoining the more distant reaction and maximum moment at a panel point when placed at that point. A uniformly distributed live load produces maximum shear in a panel when applied over the entire distance between the neutral point of that panel and the more distant reac-

tion and maximum moment at any point when applied over the entire length of the structure.

43. Maximum Moments and Shears—Structures Supported at Ends. In the preceding article moments and shears at particular sections have alone been considered, and no attention has been given to the maximum values of these functions. These maximum values must, however, be computed before the structure can be designed. For single concentrated loads and for uniform live load the value of these quantities can be easily determined as follows, for beams supported at ends.

Case 1. Maximum shear, single concentrated load, beam without floor beams. The influence line shows that the maximum value of the ordinate occurs either when $x=L$, or $L-x=L$, and equals unity, hence the maximum shear due to a load P, occurs with the load at either end of the beam. Its value equals P.

Case 2. Maximum moment on beam under same conditions as Case 1. Here the ordinate to the influence line is a maximum at the load and equals $\frac{x}{L}(L-x)$. This can be easily shown to be a maximum when $x=L-x$, hence, the maximum moment due to a load P occurs when the load is at the centre of the beam. Its value is $\frac{PL}{4}$.

Case 3. Maximum shear on same beam due to a uniform live load of intensity w. It is evident that the area between the influence line and the axis will be a maximum if section a is at either end, hence the maximum shear equals $\frac{wL}{2}$.

Case 4. Same as Case 3, but maximum moment instead of shear. The maximum moment occurs for load over entire beam, and occurs at the section where the ordinate is a maximum, which has already been shown in Case 2 to be at the centre. The moment at the centre equals $\frac{1}{8}wL^2$.

Case 5. Maximum shear. Single concentrated load. Girder with floor beams and equal panels. The maximum evidently occurs in the end panel; its value depending upon the number of panels. If n equals number of panels and P the load the maximum shear $=\frac{P(n-1)}{n}$.

Case 6. Same as Case 5, but for uniform load w per foot instead of concentrated load. Maximum shear occurs in end panels and with a load over the entire structure. Its value is $\frac{wp}{2}(n-1)$ where p = panel length.

Case 7. Same as Case 5, but maximum moment instead of maximum shear. Place load at panel point nearest centre. Maximum moment occurs at this panel point and equals $\left(\frac{P}{2}\right)\left(\frac{pn}{2}\right)$ if number of panels is even, and $\frac{Pp}{4n}(n^2-1)$ if number of panels is odd.

Case 8. Same as Case 6, but maximum moment instead of maximum shear. Maximum moment occurs at panel point nearest centre with load over entire span. Its value is $\frac{1}{8}wL^2$, when number of panels is even and $\frac{1}{8}wL^2\left(1-\frac{1}{n^2}\right)$, when number is odd.

In deriving these two quantities the following theorem may be used:

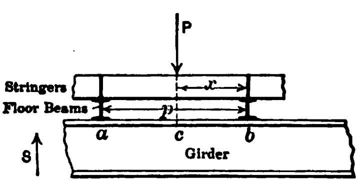


FIG. 56.

"The moment at a panel point of a girder with floor beams equals that at the corresponding point of a simple beam under the same load."

The proof of the theorem is as follows:

Let Fig. 56 represent a portion of a girder carrying floor beams.

Let M_b = moment at panel point b.

M_a = moment at panel point a.

S = shear in panel to left of given panel.

Then in accordance with rule given in Art. 34

$$M_b = M_a + Sp - P\left(\frac{x}{p}\right)p = M_a + Sp - Px.$$

This is also the value of the moment at b with the load P applied directly to the girder at the point c.

Of the formulas in this article the student is advised to

memorize that for maximum moment at the centre due to a uniform load, viz.,

$$M = \tfrac{1}{8}wL^2. \qquad . \ . \ . \ . \ . \quad (10)$$

This formula is applicable not only to simple beams, but also to girders with floor beams provided the number of panels is even.

Since the moment at a panel point equals that at the corresponding point of a simple beam under the same load, the locus of the moments at the panel points for a uniform load over the entire beam is a parabola, with a centre ordinate equal $\frac{1}{8}wL^2$, hence the ordinate at any panel point of a girder with an odd

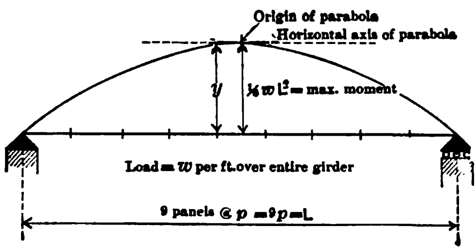


FIG. 57.

number of panels may be deduced from this value by applying the equation of a parabola. This is illustrated by Fig. 57 in which the ordinate, y, equals $\frac{1}{8}wL^2\left(1 - \frac{(\frac{1}{2}p)^2}{(4\frac{1}{2}p)^2}\right)$ = maximum moment on girder.

44. Approximate Method for Maximum Shear. In practice it is common to determine the maximum shear produced by a uniform load on an *end-supported girder with floor beams* by the following approximate but safe method.

Compute the maximum positive shear in a panel as if all panel points to right were loaded with *full* panel loads and panel points at left with no load; for maximum negative shear reverse this process.

This method is illustrated by the following example: Let the problem be the determination of the maximum positive shear in panel *cd* of the girder shown in Fig. 58 due to a uniform live load of 3000 lbs. per foot.

By the approximate method the shear should be computed for full panel loads at d, e, and f, and no loads at b and c, and will therefore equal $\left(\frac{1+2+3}{6}\right)45{,}000$ lbs. $=45{,}000$ lbs.

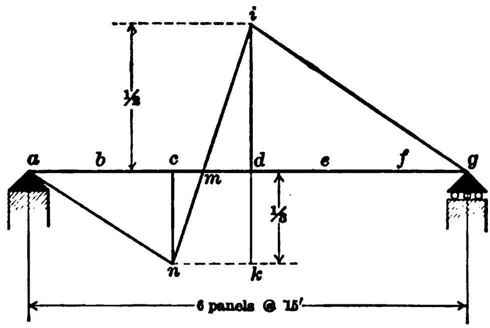


FIG. 58.

By the exact method the girder should be loaded from the right end up to the neutral point, m, in panel cd.

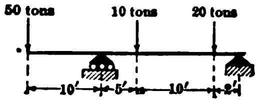
From the similar triangle of Fig. 58 it is evident that

$$\frac{md}{nk}=\frac{id}{ik}=\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}}=\frac{3}{5},$$

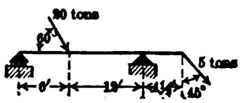
but $nk=cd=15'$. $\therefore$ $md=9'$, hence the area of the triangle $mig=54\times\frac{1}{4}=13.5$. Since the maximum shear equals the area of the triangle mig multiplied by the intensity of the load per linear foot, its value is $3000\times 13.5=40{,}500$ lbs., or considerably less than the value obtained by the approximate method, a relation which will always occur for the intermediate panels of an end supported girder. For the end panels the neutral point occurs at the end of the panel, hence for such panels the exact and approximate methods give identical results.

PROBLEMS

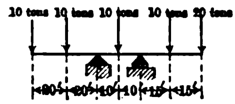
In Problems 6 to 22 inclusive, compute the horizontal and vertical components of each reaction.



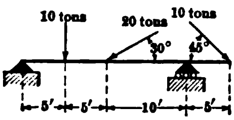
PROB. 6.



PROB. 7.



PROB. 8.



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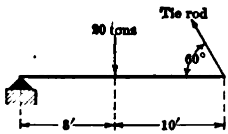
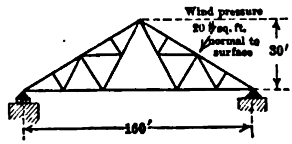
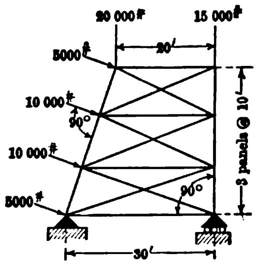


PROB. 10.



PROB. 11.

This truss is an intermediate truss of a series. Trusses spaced 20′ between centres.



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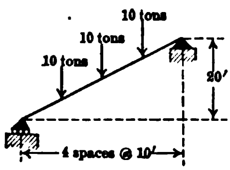

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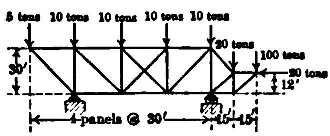

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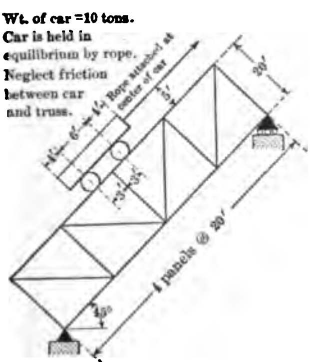

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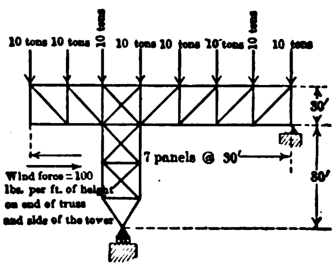

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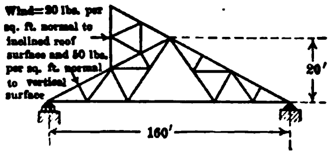
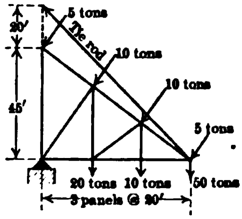


PROB. 17.

This truss is one of the end trusses of a series. Distance apart of trusses equals 20′ centre to centre.



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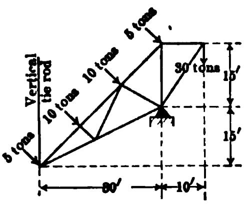
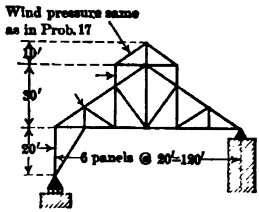
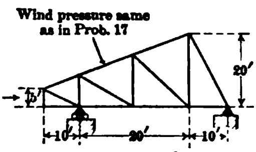


PROB. 19.



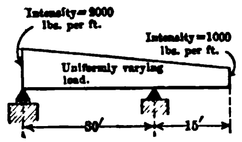
PROB. 20.

This truss is one of the intermediate trusses of a series. Distance apart of trusses equals 30′ centre to centre.

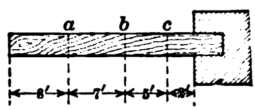


PROB. 21.

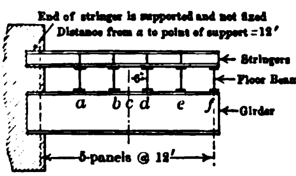
This truss is an end truss of a series. Distance apart of trusses equals 20′ centre to centre.



PROB. 22.



PROB. 23.



PROB. 24.

23. *a.* What is the magnitude of the shear at sections *a* and *c* with a concentrated load of 10,000 lbs. at *b*?

b. What is the magnitude of the shear at sections *a*, *b* and *c* with a uniform load of 1000 lbs. per linear foot over the entire beam?

24. *a.* Where should a single concentrated load be placed to cause maximum shear in panel *de*? In panel *ab*?

b. What is the magnitude of the shear at section *c* of the girder with a single concentrated load of 20,000 lbs. applied to the stringer at the centre of panel *bd*?

25. (In the following problems, relating to curves of moments and shears, and to influence lines, positive values should be plotted above the axis, and numerical values given for ordinates at all points where the curves change direction.)

Plot the curve of shears for beam shown in Prob. 23 with a uniformly varying load extending over the entire beam. Intensity of load at free end of beam 2000 lbs. per foot; at fixed end, 1000 lbs. per foot.

26. (See Prob. 24 for figure for this problem.)

a. Plot the curves of shears and moments for a uniform live load of 1000 lbs. per foot extending from the free end to the centre of panel *ab* and applied to the stringers.

b. Compare the moment at each floor beam for the loading stated in *a* with that which would exist if there were no floor beams and the same load were applied directly to the girder (i.e., a uniform load of 1000 lbs. per foot, extending 42 ft. from the free end of the girder).

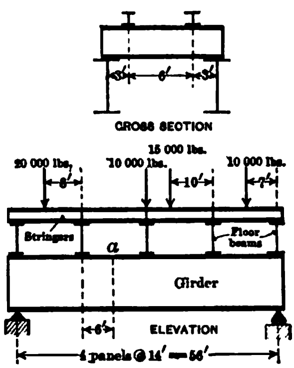
27. *a.* Draw curves of shear and moment for one girder.

b. Draw similar curves for a uniform load of 3000 lbs. per foot applied to the stringers and extending over entire span, and compare the moments at the floor beams with those which would occur at similar points if the load were applied directly to the girder.

c. Determine position of a single concentrated load for maximum shear at section *a*. For maximum moment at same section. Load to be applied at the stringers.

d. Draw the curves of dead moment and shear for following assumed weights:

Stringers, 300 lbs. per foot per stringer (this includes weight of bridge floor).



PROB. 27.

Floor beams, 100 lbs. per lineal foot of floor beam.

Girders, 200 lbs. per lineal foot per girder.

28. (See Prob. 23 for figure for this problem.)

a. Plot the influence lines for shear at sections *a* and *b*.

b. Plot the influence lines for moment at sections *a* and *b*.

29. (See Prob. 24 for figure for this problem.)

a. Plot the influence lines for shear in panel *ab* and in panel *ef* of girder. Assume girder to be directly under a stringer and load to be applied at the stringer.

b. Plot the influence lines for moment at sections *a* and *d*.

c. From an inspection of the influence line determine over what portion of the beam a uniform load should extend in order to produce maximum shear in panel *ab*, and compute the magnitude of this shear, assuming the uniform load to equal 1000 lbs. per linear foot and to be applied at the stringers.

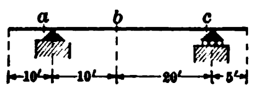
d. Same as *c*, except substitute moment at section *a* for shear in panel *ab*.

30. *a*. Plot the influence lines for shear at sections *a*, *b* and *c*.

b. Plot the influence lines for moment at sections *a*, *b* and *c*.

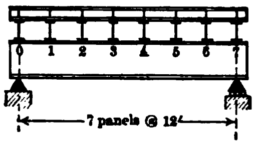
c. From an inspection of the influence lines determine where a single load should lie to give maximum shear at section *c*. To give maximum moment at section *a*.

d. From an inspection of the influence lines determine what portions of the beam should be loaded with a uniform load per foot to give maximum shear at section *c*. To give maximum moment at section *a*.



PROB. 30.

Sections *a* and *c* are to be assumed at an infinitesimal distance from the adjoining point of support.



PROB. 31.

e. Compute the maximum shears at sections *a* and *c* due to a uniform live load of 2000 lbs. per foot, and state in each case whether the shear is positive or negative.

f. Compute the maximum moments at sections *a* and *b* due to the load given in *e* and state whether positive or negative.

31. *a*. Plot influence lines for shear in panels 0–1 and 1–2. Make same assumption as to relative position of stringers and girders as in Prob. 29, and assume loads to be applied at stringers.

b. Plot influence lines for moment at sections 1 and 2.

c. From an inspection of the influence line determine where a single concentrated load should lie to cause maximum positive shear in panel 1–2 and maximum positive moment at section 2.

d. Compute by the "influence-line method" the exact maximum positive shear produced in panel 1–2 by a uniform live load of 2000 lbs. per foot, and check this result by computing the shear analytically.

e. Compute the maximum positive live shear in panel 1–2 by the approximate method given in Art. 44.

CHAPTER III

CONCENTRATED LOAD SYSTEMS

45. Shear at a Fixed Section. Girder without Floor Beams. To determine the position of loads which will produce maximum shear at a given section of a simple end-supported beam or deck-girder a method of trial may be employed. Stated briefly this consists of moving the loads along the beam by intervals corresponding to the distance between wheels and writing expressions for the change in shear thus produced. This process is repeated until the shear is found to decrease.

This method is based upon the fact that the maximum shear at a given section of a simple beam carrying concentrated loads occurs with one of the loads at an infinitesimal distance from the section.

The proof of this proposition and the application of the method to a definite case will now be given.

Loads are in thousands of lbs. per wheel.

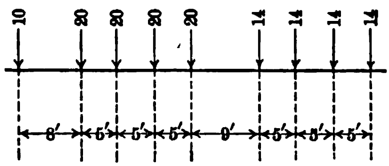


FIG. 59.

Let Fig. 59 represent a typical set of concentrated loads, in this case a single consolidation locomotive, and let it be desired to compute the maximum shear at section a, for the beam shown in Fig. 60.

The influence line for the section is shown in Fig. 60 and shows clearly that for maximum positive shear at section a most of the heavy loads must be to the right of a.

To prove that one of the loads should lie an infinitesimal distance to the right of the section, or practically *at* the section, proceed as follows: Suppose the loads to be on the beam as shown in Fig. 61.

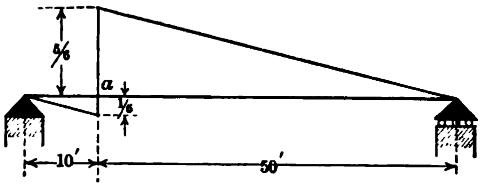


FIG. 60.

As the shear due to a set of concentrated loads in any position equals the summation of the product of the loads and their ordinates, it is evident that starting with loads in the position

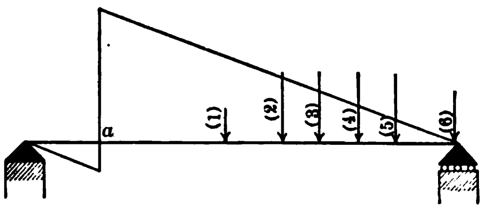


FIG. 61.

shown in Fig. 61 the shear at *a* will be increased by moving the loads to the left until load (1) reaches the section. If the loads are moved still further until load (1) passes to the left

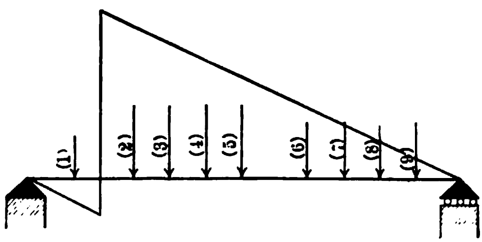


FIG. 62.

of the section there will be a sudden decrease in the shear due to load (1) crossing the section. The new position is shown by Fig. 62 from which it is again evident that if the loads be moved

still further to the left there will be an increase in shear until load (2) comes to the section, and that the result of load (2) crossing the section will be another sudden decrease in shear, after which the shear will again increase till another load reaches the section, and so on. It is also clear that the effect of a load coming on the span at the right or going off at the left during the process of moving up the loads will not affect the above conclusions.

Fig. 63 is a graphical illustration of the changes in shear at *a* of the beam shown in Fig. 60 as the loads move to the left. The ordinates represent this shear with load (1) at the point

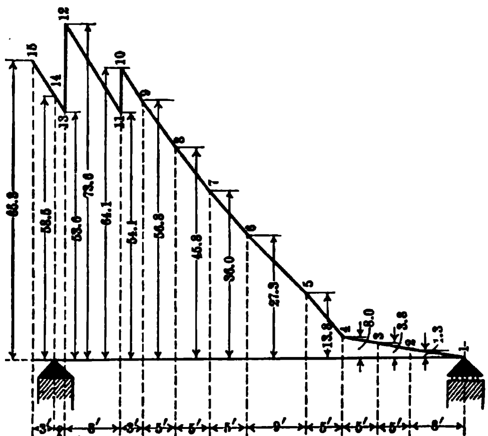


FIG. 63.

where the ordinate is shown. In consequence the line 1–2 shows the increase in shear at *a* due to moving load (1) on the span until load (2) reaches the right end; 2–3 shows the increase due to moving to the left the first two loads until load (3) reaches the right abutment, and so on up to 9–10, which shows the effect of moving the first 9 loads, i.e., all the loads, until the first load reaches the section *a*. When the first load crosses the section the shear drops suddenly by 10,000 lbs. and then increases again, as shown by 11–12, until the second load reaches section *a*. As this load crosses the section the shear is diminished by 20,000 lbs. and then increases, as shown by 13–14, until the first

load passes off the span. This does not produce a sudden change in shear but changes the slope of the line, as shown by 14–15.

From the preceding discussion it is evident that the following method may be used to determine the location of locomotive loads for maximum positive shear at any section of a simple beam:

Starting with all the loads to the right of the section and with load (1) at the section, write an expression for the change in shear due to moving load (2) to the section. If this expression shows a decrease it is evident that load (1) at the section gives the maximum shear. If, on the other hand, the expression shows an increase it will be necessary to write another expression for the increase due to moving up load (3) and so on until a decrease is finally obtained.

In practice the operation is simple, as is shown by the following example for the beam and loads of this article. It will be noticed that instead of writing an equation for the change in shear the method used is to write an inequality one side of which shows the increase in the left hand reaction due to moving up those loads *which are on the span to begin with and remain on* or which *come on during the process of moving*, and the other side of which shows the effect of a load crossing the section or going off the span to the left.

The numerical expressions for the case in question will now be given.

With (1) at section move up (2).

$$146\times\frac{8}{60}>10.$$

As the left hand quantity is greater than the right it is evident that the shear is increased by moving up load (2),

With (2) at section move up (3).

$$(146-10)\frac{5}{60}+\delta+(10)\frac{2}{60}<20.$$

Since in this case the left hand side of the inequality is less than the right hand it is evident that there is no further increase and that the maximum shear will occur with load (2) at section *a*.

As the left hand side of the above expression may not be entirely clear a few words of explanation may be added. The first term shows the increase in the left hand reaction due to

moving up those loads which are on the span to begin with and which remain on the span. The second term, δ, represents the slight increase in the shear at the section due to loads which may have come on the span at the right end of the bridge during the process of moving up the loads. This term is always small and may generally be ignored. Its value in the present case is 0. The third term gives the shear caused by load (1) when load (2) is at section a. This shear being negative and disappearing during the movement on account of the load going off the span, an increase in shear is obtained which is, therefore, placed on the left-hand side of the inequality.

Having in above fashion determined the position of the loads for maximum shear, it remains simply to compute this shear in the ordinary manner by figuring the left-hand reaction and subtracting therefrom the loads between it and the section.

46. Moment at a Fixed Section. The method of determining the position of loads for maximum moment differs somewhat from that used in determining the position for maximum shear, and is as follows:

Let the original position of the loads be as shown in Fig. 64.

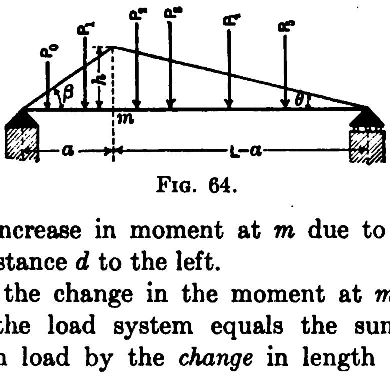


FIG. 64.

Let $\Delta M=$ increase in moment at m due to moving all the loads a short distance d to the left.

Then, since the change in the moment at m caused by the movement of the load system equals the summation of the product of each load by the *change* in length of the influence line ordinate corresponding to that load, the following expression for the increase in moment may be written:

$$\Delta M = (P_2+P_3+P_4+P_5)\ d \tan \theta - (P_0+P_1)\ d \tan \beta$$

$$= (P_2+P_3+P_4+P_5) d \frac{h}{L-a} - (P_0+P_1) d \frac{h}{a}$$

$$\therefore \qquad \frac{\Delta M}{hd} = \frac{P_2+P_3+P_4+P_5}{L-a} - \frac{P_0+P_1}{a}.$$

This equation shows that the moment at m will be increased by moving the loads to the left if the *average load per foot* on the *right* of m be greater than the *average load per foot* on the *left*. The converse of this proposition is also true. It should be noted that if the average load per foot on the right equals the average load per foot on the left there will be no change in moment caused by moving the loads.

The above equations are true, provided the relative position of the loads does not change; that is, if no load comes on from the right, or goes off to the left, or passes the section. It may be readily seen, however, that if the average load per foot on the right exceeds that on the left a movement to the left sufficient to bring another load on from the right or to cause a load to go off to the left serves to increase the moment more rapidly, and hence does not vitiate the conclusion that the loads should be moved to the left. It is also evident that the movement to the left should be continued until P_2 reaches the section, hence we have the following theorem:

For maximum moment at any section one load must lie at the section, and the loads must be so located that with that load just to the right of the section the average load per foot on the right is greater than that on the left, while with that load just to the left of the section the average load per foot on the left is greater than that on the right.

A special case of the above is when the average load per foot on one side equals the average load per foot on the other side. In this case a load does not have to lie at the section, but if it does lie at the section the moment will be equal to the maximum, hence the theorem applies for this case also.

It should be noticed that the proof of this theorem would be equally applicable to any case where the influence line is composed of two straight lines, and that in consequence the theorem is very useful for many cases other than that of moment on a simple beam.

The application of this theorem is simple, but it sometimes happens that several loads of the same system will be found to satisfy the above criterion. This is explained by the fact that a different set of loads may be on the span for each position, and consequently there may be several maxima. In such cases it is usually necessary either to compute the value of each

maximum, or else to compute the change in moment due to moving the loads from one maximum position to another.

A numerical example of the determination of the position for maximum moment will now be given.

Let the loads and span be as in Art. 45, and let the problem be to find the position giving maximum moment at *a*.

First try load 2:	Av. load per ft. on left.		Av. load per ft. on right.
Load (2) to right of section............	$\frac{10}{10}$	$<$	$\frac{136}{50}$
Load (2) to left of section.............	$\frac{30}{10}$	$>$	$\frac{116}{50}$

Load (2) gives a maximum:

Try load (3):	Av. load per ft. on left.		Av. load per ft. on right.
Load (3) to right of section	$\frac{20}{10}$	$<$	$\frac{116}{50}$
Load (3) to left of section	$\frac{40}{10}$	$>$	$\frac{96}{50}$

Load (3) also gives a maximum.

It is seen by inspection that in this case it is unnecessary to try load (4) and that loads (2) and (3) are the only ones giving maximum moments. To determine which of these is the larger it is advisable to compute both independently and then check the results by computing the change in moment produced by starting with load (2) at *a* and moving load (3) to *a*.

That the maximum moment at a given section due to a set of concentrated loads always occurs with a load at the section is apparent from the fact that the maximum moment for a given position of loads occurs where the shear curve crosses the axis; i.e., where the shear equals or passes through zero, and that this can never happen except at one of the loads.

47. Shear. Girder with Floor Beams. For such girders the maximum shear in every panel must be computed. The method of determining the position of the loads differs in detail from that given in Art. 45, although the same general method may be used.

To illustrate this case the bridge shown in Fig. 65 is chosen. Here again, for greater clearness, the stringers and floor beams

are shown above the girders, though, as already explained, such construction is uncommon. End floor beams are also used, but this makes no difference in the method or its application.

Consider first the position of loads for shear in panel 1. In this case it is clear that the maximum shear occurs with the same condition which would produce maximum moment at panel point a, since the proof given in Art. 46 applies equally well here. In consequence, one of the loads must lie at a. To determine which, either the method of moving up the loads explained in Art. 45 may be used, or that of obtaining the position of the loads for maximum moment at a. If the latter plan be adopted it may happen that more than one position will be found to give a maximum, and hence an extra computation will be needed. This latter is, however, useful as a check and is not a conclusive argument against the method since an approximate check computation with another load at the section should invariably be made.

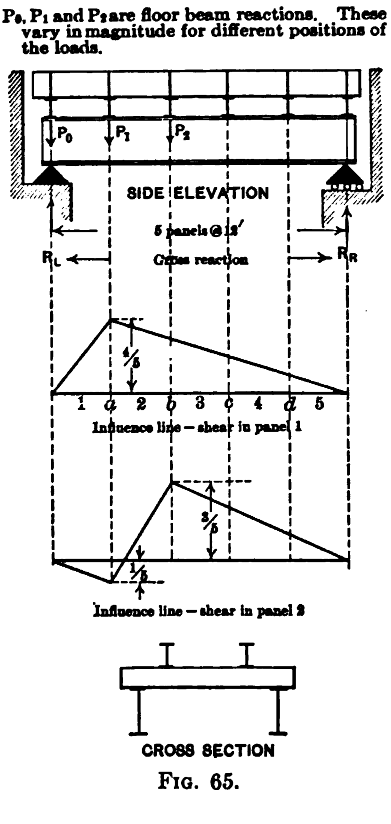


FIG. 65.

The fact that the maximum shear in the end panel and the maximum moment at the first panel point occur simultaneously is important. It follows that since none of the live loads can be applied to the girder between panel points, the maximum live moment at the first panel point equals the product of the maximum live shear in the end panel and the length of that panel.

For intermediate panels the latter method can not be used since it is incorrect, except for cases where the influence line

is composed of two straight lines forming a triangle with the axis of the beam, a condition which does not occur for intermediate panels. For such panels therefore the method of Art. 45 will be adopted. Examination of the influence line shown in Fig. 65 for the shear in the second panel, which is typical of the influence lines for all intermediate panels, shows that the loads when brought on from the right must at least be moved to the left until the first load reaches *b*. Further movement to the left will cause a decrease in the shear due to the first load, but an increase due to the loads on the right. If the result is a net increase, the loads should be moved until load (2) reaches *b*. This conclusion is uninfluenced by the action of other loads which may come on the span from the right, or by the fact that load (1) may pass *a*. Further movement to the left produces an additional increase in shear due to loads to the right of *b*, but a decrease due to load (2), and either an increase or a decrease due to load (1). If the expression for the change is positive it will remain so until load (3) reaches the section unless load (1) passes off the bridge, which would lower the rate of increase and perhaps cause a decrease. This condition is, however, not likely to occur and may be neglected.

In view of the foregoing it may be stated that for maximum positive shear in either end or intermediate panels, one of the loads must lie at the panel point to the right.

Before proceeding to a numerical illustration of these principles, the student should observe that the increase in shear in panel 1 equals the increase in R_L minus the increase in P_0; that the increase in shear in panel 2 equals the increase in R_L minus the combined increase in P_0 and P_1, and similarly for other panels.

A load which passes off the span in the process of moving up should always be considered by itself. It should be noted that the change of shear, or of any other function, due to removing a load from a structure, is equal to the shear or other function caused by the load when on the structure. Hence, to find the change in shear due to a load passing off the span compute the shear due to it in its original position before the loads are moved.

The application of these principles to the structure shown in Fig. 65 will now be given for the locomotive shown in Fig. 59.

Shear in End Panel. Method of Moving-up the Loads.

Start with load (1) at panel point a.

	Increase in R_L		Increase in P_0
Move load (2) to a,	$146\times\frac{8}{60}+\delta$	$>$	$10\times\frac{8}{12}$
$\therefore$ shear is increased.	$\delta=0$ in this case.		
Move load (3) to a,	$136\times\frac{5}{60}+\delta$	$>$	$20\times\frac{5}{12}+10\times\frac{4}{12}\times\frac{4}{5}$.*
$\therefore$ shear is increased.	$\delta=0$ in this case.		

The fact that the increase in moving up load 3 is very slight and that the next step of moving up load 4 would materially increase the change in P_0 without increasing that in P_1 makes it evident that load (3) at section gives the maximum shear.

The value of the maximum shear in the end panel may now be computed. The expression for it is

$$\frac{20}{60}(53+48+43+38)+\frac{14}{60}(29+24+19+14)-20\times\frac{5}{12}=73$$

(thousand lb. units.)

To show that above conclusions are correct the shear with load (2) at a will be computed.

$$\frac{20}{60}(48+43+38+33)+\frac{14}{60}(24+19+14+9)+10\times\frac{4}{12}\times\frac{4}{5}=72.07$$

(thousand lb. units.)

The value of this is less than that for load (3) at section and is therefore in accordance with the conclusions of the previous method.

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* The last term in the above expression gives the shear due to load (1) when load (2) is at a. Its value is obtained by computing the floor beam reaction P_1 and the shear due to it. The reaction P_0 may be ignored since it produces no shear in the girder. The same result should be obtained by the usual method of computing R_L and subtracting P_0 from it; this gives $\left(\frac{56}{60}-\frac{8}{12}\right)10$, which equals the value already found.

Shear in End Panel. Average Load Method.

	Av. load per ft. on left.		Av. load per ft. on right.	
Load (2) to right of panel pt. a,	$\frac{10}{1}$	$<$	$\frac{136}{4}$	$\therefore$ Load (2) gives a max.
Load (2) to left of panel pt. a,	$\frac{30}{1}$	$>$	$\frac{116}{4}$	
Load (3) to right of panel pt. a,	$\frac{20}{1}$	$<$	$\frac{116}{4}$	$\therefore$ Load (3) gives a max.
Load (3) to left of panel pt. a,	$\frac{40}{1}$	$>$	$\frac{96}{4}$	
Load (4) to right of panel pt. a,	$\frac{40}{1}$	$>$	$\frac{96}{4}$	$\therefore$ Load (4) does not give a max.

From these expressions it is seen that by the application of the average load criterion, loads (2) and (3) are found to give maxima, and that it is necessary to calculate both to determine the greater.

It should be noted that in the application of the average load method the average shear per panel instead of the average shear per foot has been used. This is simpler and gives the same result when the panels are of equal length as in the bridge under consideration. If the panels are of unequal length this method would be *incorrect.*

Shear in Second Panel. Method of Moving up the Loads.

Start with load (1) at panel point b.

	Increase in R_L		Increase in (P_0+P_1)	
Move load (2) to b,	$104\times\frac{8}{60}+\delta$	$>$	$10\times\frac{8}{12}$	$\therefore$ Shear is increased

$\delta=14\times\frac{9}{60}$ but it is evident that this value need not have been computed since it is too small to alter the result.

Move load (3) to b,

Increase in R_L		Increase in (P_0+P_1)	
$132\times\frac{5}{60}+\delta$	$<$	$10\times\frac{4}{12}+20\times\frac{5}{12}$	$\therefore$ Load (2) gives a maximum

$\delta=14\times\frac{2}{60}$ (necessary to compute in this case since otherwise results would be doubtful).

The right-hand side of above inequality may require some explanation.

$$\text{With load (2) at } b,\ P_0=0 \qquad \text{and} \qquad P_1=\frac{8}{12}\,10.$$

$$\text{With load (3) at } b,\ P_0=\frac{1}{12}\,10 \qquad \text{and} \qquad P_1=\frac{11}{12}\,10+\frac{5}{12}\,20.$$

$$\therefore \quad \text{Increase in } (P_0+P_1)=10\times\left(\frac{12}{12}-\frac{8}{12}\right)+\frac{5}{12}\,20.$$

That is, the increase in (P_0+P_1) when load (1) is moved from the second into the first panel, equals the reaction on the floor beam at b due to load (1) when load (2) is at b, plus the increase in the reaction on the floor beam at a due to moving the second load into the second panel.

The value of the maximum shear in the second panel equals

$$10\times\frac{44}{60}+\frac{20}{60}(36+31+26+21)+\frac{14}{60}(12+7+2)-\frac{10\times 8}{12}=43.56$$

(thousand-lb. units).

As an approximate check the corresponding shear with load (3) at b has been computed and found to be 43.37. If the increase in shear as determined from the expression for the increase due to moving up load (3) be added to this the result should equal the shear with load (2) at b, thus giving an exact check.

The student will observe that in all cases where no load goes off or comes on the span, or goes out of the panel, the distance which the loads are moved appears on both sides of the inequality and may be omitted. Moreover, the denominator of the left-hand term equals the span length and that of the right-hand term the panel length. Hence we may say that for the special conditions noted the shear will be increased by moving up the loads whenever the average load per foot on the entire span exceeds that on the given panel.

48. Formula for Position of Loads for Maximum Shear for Intermediate Panels. *Girder with Floor Beams.* The method of moving up the loads as used in the preceding article is simple and very general. It is applicable not only to the determination of the position of loads for maximum shear but to the determination of position for many other functions. The student should understand it thoroughly and apply it to many different

cases until he thoroughly comprehends the influence of such load systems upon the various portions of the girder.

For the practitioner who may wish a definite formula for determining the position, the following may be of use for *intermediate panels*.

(*c*) The maximum shear will occur with the first load of the system at the right end of the panel if $\Sigma\frac{Pa}{L} \overline{\gtrless} \Sigma\frac{P_1 a_1}{p}$. Apply the same criterion to the second load and to following loads if necessary until the inequality is satisfied. In the above inequality, P = any load which may be on the span or which may come on during the moving up of the loads from one position to another, a = distance which that load is moved. P_1 = any load which may be at any time in the panel under consideration during the process of moving the loads. a_1 = the distance which P_1 moves in the panel.

If no load comes on or goes off the span and if no load passes out of the panel, $a = a_1$, and we may write

$$\Sigma\frac{P}{L} \overline{\gtrless} \Sigma\frac{P_1}{p}.$$

It follows that for this case the first load should lie at the panel point unless the average load per foot on the entire span is greater than that on the given panel. In this latter case the second load should be tried at the panel point and so on until the position for maximum shear is determined.

49. Maximum Moment. *Girder with Floor Beams.* For girders with floor beams it is customary to compute maximum moments at panel points only. If, for any reason, the maximum moment between panel points is desired it may be obtained with sufficient accuracy by interpolation.

For uniform live loads and for concentrated loads which are fixed in position interpolation gives exact results since the curve of moments for such loads consists of a series of straight lines. The same is also true for moments due to the weight of the floor system, but is slightly in error for the weight of the girder itself. For a system of moving loads this method is somewhat inaccurate but is on the safe side, and hence may be used with security. This is shown by the following demonstration which refers to Fig. 66.

Let the ordinate bA represent the maximum live moment

at any panel point, b, due to a concentrated load system. For the position producing this maximum the moment curve for the portion of the girder between b and c will be the line AB, Bc representing the moment at c for the position of the loads giving maximum moment at b. If the loads be now moved so as to give a maximum moment at c we shall have cB' and bA' as the ordinates for moments at c and b respectively for this new position, and $B'A'$ will be the moment curve between b and c. It is evident from the figure that interpolation between the maximum moment at b and that at c will give a safe value for the maximum moment at any point in the panel, since the line AB can never rise above AB' nor the line $B'A'$ above $B'A$; therefore, the ordinate xx' for the moment at x can never be less than the actual maximum moment at x. It will readily be seen by drawing an influence line for the moment at x that for maximum moment some load should lie at either panel point b

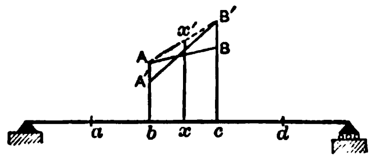


FIG. 66.

or c; that the moment at c with the loads in the position necessary for maximum moment at b can never exceed the maximum moment at c and will almost invariably be less than that; and that this principle holds good for the condition when the moment at b is a maximum. This proof is perfectly general and applicable to any panel.

50. Moment and Shear at the Critical Section. The cases already treated have been for shear and moment at stated sections of simple beams and for panels and panel points of girders with floor beams. For the latter it is necessary and sufficient to compute the maximum shear in each panel and the maximum moment at each panel point, since thereby the maximum values of these functions will be obtained. For beams or girders which are not loaded by means of floor beams it is always necessary to compute maximum values of shears and moments, and in addition, for long girders, the values at intermediate sections taken with sufficient frequency to insure a good design.

In order to determine the maximum values it is necessary to locate the sections at which they occur, that is, the *critical sections*.

For shear the critical section is an infinitesimal distance from one of the points of support. This needs no demonstration, as an inspection of influence lines for various sections including one at the end furnishes sufficient proof.

For moment with uniform load it has already been shown that the maximum moment occurs at the centre and equals $\frac{1}{8}wL^2$, when w equals the load per foot and L the span.

With a system of concentrated loads the maximum moment may not occur at the centre though the critical section will be very near the centre. To treat this case it is necessary to

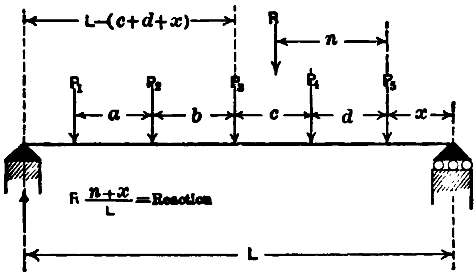


FIG. 67.

make use of the already established principle that for maximum moment at any section of a beam under a system of concentrated loads one of the loads must lie at the section. If, therefore, it is possible to determine the location of the system of loads as they cross the span such that the moment at any one load is a maximum, the problem can be solved by trying a sufficient number of loads and computing the different maxima. As will appear later the critical section is always near the centre of the span, hence, as a rule, only loads need be tried which are found to give a maximum moment at the centre.

Consider the set of loads shown in Fig. 67, and let the problem be the determination of the position of these loads in order that the moment at P_3 may be a maximum. Let R be the resultant of the loads P_1 to P_5 inclusive and n its distance from the last load P_5.

Let x be the distance from P_5 to the right support when the loads lie in the proper position for maximum moment at P_3.

Then the moment at P_3 is given by the equation

$$M_3 = R\frac{(n+x)}{L}[L-(c+d+x)] - P_1(a+b) - P_2 b.$$

For maximum value of M_3 differentiate with respect to x and put the first derivative equal to 0. This gives

$$\frac{dM_3}{dx} = \frac{R}{L}[-n+L-c-d-2x] = 0.$$

Therefore, in order to find the maximum moment at P_3 as the loads cross the span, P_3 must be so located that

$$-n+L-c-d-2x=0$$

or
$$L-(c+d+x)=n+x.$$

That is, the resultant of the loads on the span when the maximum moment at P_3 occurs must lie as far from the right support as the load itself lies from the left support, or in other words the *centre* of the *span* must lie *half way* between the *resultant* and the *load*.

The following examples serve to illustrate the application of this principle:

Problem. Compute the absolute maximum moment on a simple beam of 12-ft. span due to two wheel loads of 10,000 lbs. each spaced 6 ft. between centres.

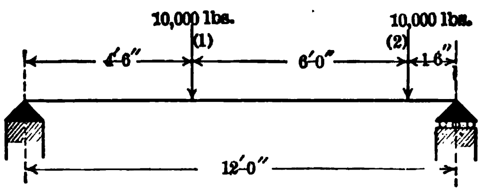


FIG. 68.

Solution. In this case there are two equal loads, hence it is immaterial which load is considered. For maximum moment at load (1) the loads should lie as shown in Fig. 68, the centre of the span being half way between load (1) and the resultant of the two loads. The moment at the first load will then equal

$$20{,}000\frac{(6-1\frac{1}{2})^2}{12} = 33{,}750 \text{ ft.-lbs.},$$

The maximum moment at the centre for this beam would be 30,000 ft.-lbs., hence the absolute maximum moment exceeds the maximum centre moment by over 10 per cent.

It should be particularly noted that the demonstration which has been given only serves to fix the position for maximum moment at a given load with certain assumed loads on the span, and that if a different set of loads be on the span the position will be different. To illustrate this consider the same loads as in the previous examples and a span of 10 ft. There are then two positions of the first load which give maximum moment. First, assume only the first load to be on the span; in this case it should be placed at the centre and the moment would be 25,000 ft.-lbs. Second, assume two loads on the span; in this case the centre of the span should be half way between the resultant of the two loads and the first load, and the maximum moment at the first load will equal.

$$20{,}000\frac{(5-1\frac{1}{2})^2}{10}=24{,}500 \text{ ft.-lbs.},$$

which is somewhat less than with one load at the centre. In such a case the length of span can easily be determined for which one load at the centre gives a moment at the load equal to that with two loads on the span. In general it is necessary to consider both cases when dealing with two loads.

The absolute maximum moment on spans above 25 or 30 ft. in length does not materially differ from the maximum centre moment, and in practice the latter is generally used.

For the loads previously considered with a 30-ft. span the absolute maximum moment $=20{,}000\frac{(15-1\frac{1}{2})^2}{30}=121{,}500$ ft.-lbs., while maximum centre moment $=20{,}000\left(\frac{12}{30}\right)15=120{,}000$ ft.-lbs. The difference is about one per cent, which is so small as to be negligible.

The following example serves to show the application of this principle for a locomotive loading:

Problem. Determine the maximum moment on a simple beam of 21-ft. span due to the locomotive given in Art. 45.

Solution. First determine which load or loads give maximum moment at the centre, as it is probable that one of these loads will give the

absolute maximum moment. By applying the criterion for maximum moment, loads (3) and (4) are found to give maxima, but it is clear that the centre moment with load (3) at the centre will equal the centre moment with load (4) at the centre, and that it makes no difference whether we use one or the other load. Let the maximum moment therefore be determined at load (3), assuming loads (2–5) inclusive on the span. The position for maximum moment will then be as shown in Fig. 69 and the moment at load (3) will equal

$$80{,}000\frac{(10\frac{1}{2}-1\frac{1}{4})^2}{21}-20{,}000\times 5=225{,}950 \text{ ft.-lbs.}$$

In this case it is impossible to get more than four loads on the span at once. If three loads are on the span the resultant coincides with load (3), hence for a maximum for this assumption load (3) should lie at the centre, but this is inconsistent with three loads being on

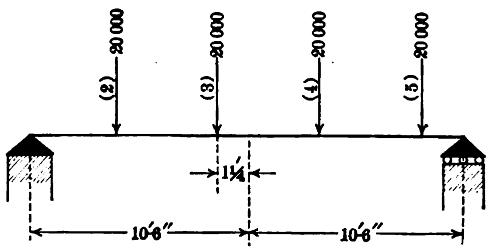


FIG. 69.

the span, hence a maximum at load (3) with only three loads on the span cannot be obtained, and the case considered gives the absolute maximum moment.

The maximum centre moment for these loads occurs with either load (3) or load (4) at the centre and equals

$$80{,}000\times\frac{8}{21}\times 10\frac{1}{2}-20{,}000\times 5=220{,}000 \text{ ft.-lbs.,}$$

so that in this case the difference is only 2.7 per cent.

51. Moments and Shears. *Floor Beams and Transverse Girders.* As a preliminary step in the examination of this case the influence lines shown in Figs. 70 and 71 have been drawn. These are influence lines for stringer reactions on floor beams. Since the stringers are simple beams of a length equal to one panel and are supported at the ends upon the floor beams, it is evident that a load moving along the bridge causes no reaction on a floor beam unless it is on the stringers in one of the panels

adjoining the floor beams in question. Fig. 70 represents the stringer reactions on an intermediate floor beam and Fig. 71 on an end floor beam.

It will be noticed that the influence line shown in Fig. 70 has the same characteristics as the influence line for moment

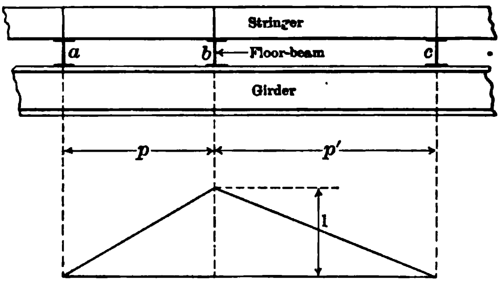


FIG. 70.—Influence line for stringer reaction on floor beam at *b*.

at any section of a simple end-supported beam, hence the demonstration of Art. 46 is applicable. The conclusion may therefore be at once drawn that for maximum reaction on an intermediate floor beam one load must lie at the beam, and that load must be one which, when placed just to the right of the given floor

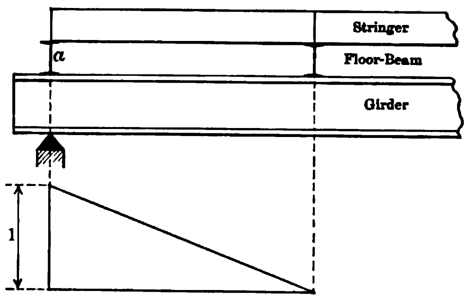


FIG. 71.—Influence line for stringer reaction on floor beam at *a*.

beam, makes the average load per foot on the stringers in the right hand panel greater than on those in the left panel, and when placed just to the left of the floor beam reverses this condition. For the end floor beam, the maximum reaction occurs for the loading giving maximum stringer reaction and equals that reaction.

It remains to consider the actual moments and shears on

the floor beams. Curves of moments and shears for a floor beam due to stringer reactions are shown in Fig. 72.

Both shear and moment are direct functions of the stringer reactions. The maximum moment must occur at one of the stringers, since the floor beam is in the condition of a girder loaded with concentrated loads and the curve of shears can cross the axis only at a load. The case illustrated is not the usual one, since the stringers are unsymmetrically placed with respect to the centre of the floor beam. Were the floor beam symmetrical the maximum moment would occur at both stringers and at all points between. Since, in the actual design the dead

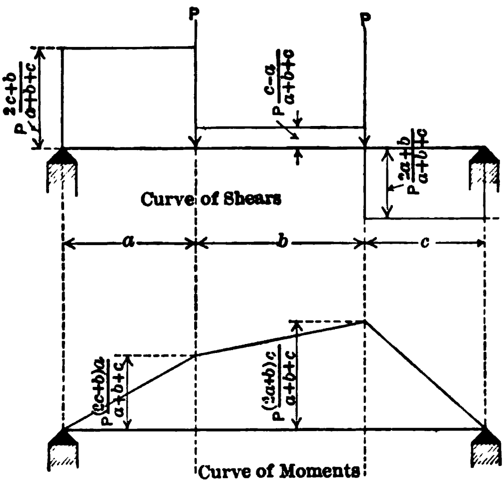


FIG. 72.

load of the floor beam would also have to be considered, the maximum combined live and dead moment for the ordinary symmetrical floor beam occurs at the centre.

For floor beams where the stringers in one of the adjoining panels are not of equal length, that is, where the panel is a skew panel, special treatment is necessary. It is usually advisable to treat this case by means of influence lines without attempting to apply special rules.

The application of the methods just given to the determination of the maximum moment and shear on a floor beam (or a transverse girder, such as a cross girder in an elevated railroad structure), will now be illustrated.

Problem. Determine maximum moment and shear on floor beam b of Fig. 73 for loads shown in Fig. 59, Art. 45.

It may be easily seen that a given load when equidistant from the floor beam b produces greater reaction if on the longer stringer, hence it is probable that the maximum reaction in this case will occur with the

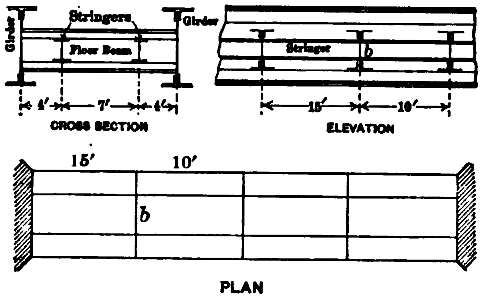


FIG. 73.

greater number of loads on the 15-ft. panel. Let the loads, therefore, be brought on from the left.

	Av. load per ft. on left.		Av. load per ft. on right.	
Load (2) to left	$\frac{60}{15}$	>	$\frac{10}{10}$	Load (2) does not give a maximum.
Load (2) to right..............	$\frac{60}{15}$	>	$\frac{30}{10}$	
Load (3) to left	$\frac{60}{15}$	>	$\frac{20}{10}$	Load (3) gives a maximum.
Load (3) to right..............	$\frac{40}{15}$	<	$\frac{40}{10}$	
Load (4) to left	$\frac{54}{15}$	<	$\frac{40}{10}$	Load (4) does not give a maximum.

Load (3) at b evidently gives the maximum floor-beam reaction. Its value is given by the expression,

$$20\times\frac{5+10}{10}+20\frac{10+5}{15}=50,$$

that is, the reaction on the floor beam at each stringer connection equals 50,000 lbs.

The floor beam is then in the condition shown by Fig. 74.

The maximum shear $=$ 50,000 lbs. and the maximum moment $=$ 200,000 ft.-lbs.

Before concluding this article the beginner should be cautioned to avoid the mistake that is frequently made of adding the maximum live reactions on two adjoining stringers to determine the floor beam load at a point such as m in Fig. 74. The fact that the maximum reaction on a stringer occurs when one of the heavy loads lies at the end of the stringer is sufficient to show that the same condition cannot exist on the adjoining stringer in the next panel, because such a condition would necessitate two wheel loads occupying practically the same place at the same time.

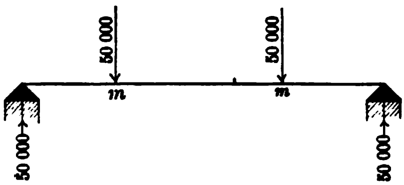


Diagram showing loads on floor beam.

FIG. 74.

52. Moment Diagram. To save repetition of computations for a given set of concentrated loads when used for varying spans, it is customary to use a moment diagram upon which certain quantities frequently required and unaffected by the length of spans are placed once for all. Upon this diagram the loads are plotted to a convenient scale at top and bottom of sheet for convenience in reading, and the quantities desired are placed between. The diagram is used in connection with another sheet upon which the span is drawn to the scale used in plotting the loads. The diagram shown in Fig. 75 is of a convenient form and is self-explanatory.

The use of the diagram can be readily understood from the simple example that follows. Let the problem be the computation of the moment at the second panel point from the left of a span having 5 panels at 20 ft. when load (8) is at the given panel point. Place the plotted span on a separate sheet so that load (8) is over panel point 2; the ends of the span will then be in the positions shown at bottom of Fig. 75. Since the desired moment equals the moment of the left reaction due to loads (2) to (17) inclusive about load (8), minus the moment of loads (2) to (7) inclusive about the same point, it is necessary

MOMENT DIAGRAM—COOPER'S E_{40} LOCOMOTIVE

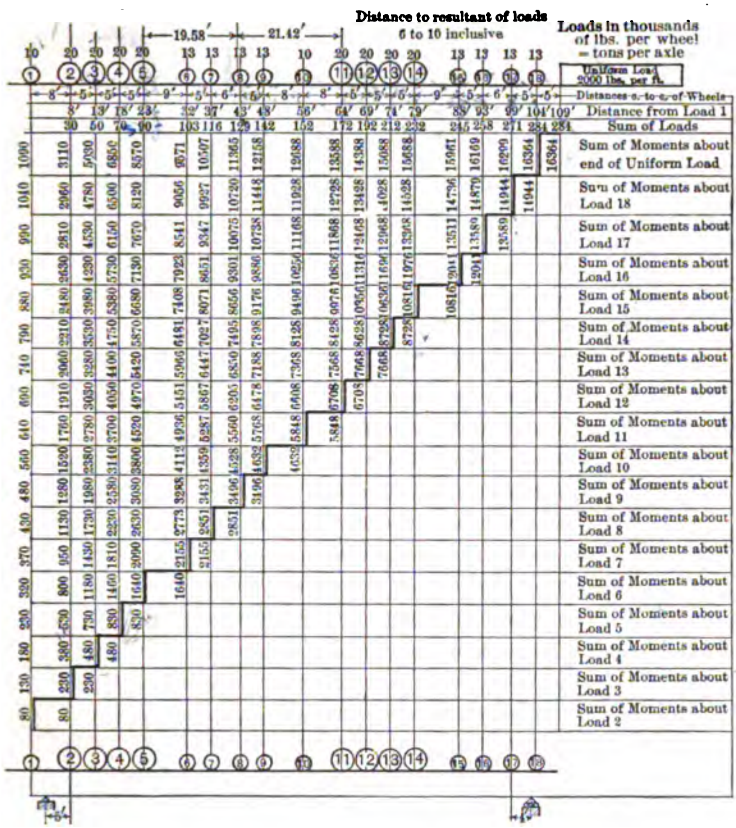


FIG. 75.

Moments are in thousands of foot-pounds per rail and are the summations of the moments of all the loads to the left of and including that under which they appear about the load indicated in right-hand columns; e.g., the value 10256 under load 10 and opposite load 16 in right-hand column is the moment about load 16 of the loads 1 to 10 inclusive.

Moments to right of zig-zag line are moments about load under which they appear of all loads to left thereof.

Application of Diagram. To find moment about a given load, say load 14 of a certain number of loads to left, say 6 to 13 inclusive, proceed as follows: Find in diagram, moment about 14 of all loads to left; subtract from it moment about 5 of all loads to left of 5 plus the product of the sum of loads 1 to 5 inclusive and distance from load 5 to load 14. The result will be the desired moment. The expression thus obtained $=8728-[830+90\times 56]$ $=2858$. The moment of the same loads about a point between 14 and 15 and distant x ft. from 14 can be found by adding to quantity just obtained the product of the sum of the loads 6 to 13 and the distance x. See Article 34.

to compute these two quantities. The moment of the left reaction equals the moment of the loads on the span about the right reaction divided by 5 panel lengths and multiplied by 2 panel lengths.

This equals $\frac{2}{5}[13,589-990+(271-10)\times 4]=\frac{2}{5}13,643=5457.2.$

The moment of loads (2) to (7) inclusive about panel point $2=2851-10\times 43=2421.$

Hence, the moment desired $=5457.2-2421=3036.2$ expressed in units of thousands of pounds per rail or tons per track.

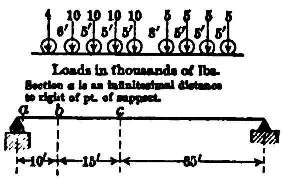
This method involves the application of the principle stated in Art. 34, which should be thoroughly understood.

PROBLEMS

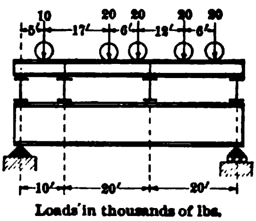
32. *a.* Compute the maximum positive shear in thousands of pounds at sections *a*, *b* and *c* for the system of moving loads shown.

b. Compute maximum moment in foot-pounds at *b* and *c* for the system of moving loads.

c. Compute uniform live load per foot which would give a maximum moment at section *b* equal to that found for the system of moving loads.

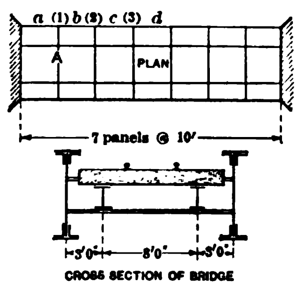


PROB. 32.



PROB. 33.

33. Draw curves of shear and moment for one girder for the concentrated loads when placed in the position shown in the figure. (Bridge has two girders located symmetrically with respect to the loads.)



PROB. 34.

34. *a.* Draw influence lines for shear in panels *a* and *b* of one girder. Assume two loads of unity to pass over the structure—one on each rail.

b. Compute maximum dead shear in panels *a*, *b* and *c* of one girder, and dead moment at panel points 1, 2 and 3 for the following dead weight:

Stringers, 100 lbs. per foot per stringer.
Track (rails, ties, etc.), 400 lbs. per foot per track.
Floor beams, 100 lbs. per foot per floor beam.
Girders, 300 lbs. per foot per girder.

c. Draw curve of dead moments for floor beam *A*, using above dead weights.

35. *a.* Determine the position for maximum positive shear in panels *a* and *b* of bridge in Prob. 34.

1. For the system of concentrated loads shown in Fig. 11, coming on from right.
2. For the same loads with train running in opposite direction, i.e., coming on from left.
3. In panel *a* for one of the locomotives shown in Fig. 11 followed, at a distance of 5 ft., by the uniform load, the loads coming on from right.

b. Compute the live shear in panels *a* and *b* for each of the positions previously determined.

c. Determine the position of the concentrated load system for maximum moment at panel point 2, considering only the first and third cases given under *a*. Try driving wheels only.

d. Compute the live moment at panel point 2 for each of the positions previously determined.

e. Compute the maximum live moment at panel point 1 for the system of concentrated loads previously used.

f. Compute the uniform live load per foot which will give a live shear in panel *b* equal to 93,750 lbs.

36. Compute for the bridge of Prob. 34 the maximum live shear and moment on floor beam *A*, using same loads as in Prob. 35.

CHAPTER IV

BEAM DESIGN

53. Formulas. In order to determine the proper size of beams required to carry given external bending moments and shears, it is necessary to make use of formulas expressing the relation between the outer and inner forces. Such formulas are deduced in all standard books on mechanics, and are as follows for beams of homogeneous material and of ordinary proportions:

$$M=\frac{fI}{y}, \qquad (12)$$

$$v=\frac{VQ}{bI}. \qquad (13)$$

The terms in these formulas are as follows for any cross-section of the beam:

M = external bending moment at section in inch-pounds.

I = moment of inertia in inches^4 about the neutral axis of the section.

y = distance in inches from neutral axis to any fibre.

f = direct fibre stress at distance y from neutral axis.

Q = statical moment[1] about the neutral axis of the cross-section of that portion of the section lying either above or below an axis parallel to the neutral axis and at distance y from it.

v = intensity of the longitudinal shear per square inch along any plane parallel to the neutral plane of the beam and at distance y from it.

V = external shear in pounds at the section.

b = width of beam at distance y from neutral axis.

[1] Statical moment of a given area about any axis equals the area multiplied by the distance from its centre of gravity to that axis.

The application of these formulas to actual problems of design requires the selection of beams such that at no section shall $\frac{My}{I}$ exceed the maximum allowable value of f, nor $\frac{VQ}{bI}$ exceed the allowable value of v.

It is evident that under all conditions f will attain a maximum value for any given cross-section at the fibre farthest removed from the neutral axis, since y will then be a maximum. For beams of uniform width the largest value of v for any given section will occur at the neutral axis since the statical moment Q has its maximum value about the neutral axis and b is constant. The absolute maximum values of f and v for a beam are functions of the external moment and shear, and of the cross-section of the beam. If the beam is of constant cross-section throughout, then these maximum values will occur at the section where M and V respectively have maximum values.

In a beam of variable section, f and v may attain maximum values at several points. For greatest economy of material the maximum values of f and v for the different cross-sections of the beam should be constant throughout its length, but it is seldom or never attempted to obtain this conditions, since the additional *labor* cost would far exceed the saving due to *economy of material.*

Substituting limiting values in formula (12), the following working formula is obtained:

$$M=\frac{sI}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

in which s = maximum allowable working value of f in lbs. per sq. inch.

c = distance in inches from neutral axis to the extreme fibre of any given section,

M = maximum allowable bending moment on the beam in inch lbs.

For beams of rectangular cross-section having a height h and a width b formula (14) becomes

$$M=\frac{s\left(\frac{1}{12}bh^3\right)}{h/2}=\frac{1}{6}sbh^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The maximum value of v for beams of rectangular cross-section is given by formula (16) in which A = area of the cross-section:

$\frac{QV}{I.b}$

$$v = V\frac{\left(b\frac{h}{2}\right)\frac{h}{4}}{b\cdot\left(\frac{1}{12}bh^3\right)} = \frac{3V}{2bh} = \frac{3V}{2A}, \quad . \quad . \quad . \quad . \qquad (16)$$

54. Method of Design. Frequently the design of beams requires merely the application of formula (14), and the shearing strength need not be considered. In the case of comparatively short beams, however, the shearing strength is more important and should be investigated. In wooden beams this is especially important, since the resistance of wood to longitudinal shear is small and such beams may fail by splitting longitudinally. The design of reinforced concrete beams also requires the application of formula (13).

55. Wooden Beams. In selecting wooden beams care should be taken to use commercial sizes only. The following table gives such sizes:

Spruce:

2 × 3, 2 × 4, 2 × 5, 2 × 6, 2 × 7, 2 × 8, 2 × 10, 2 × 12.
3 × 4, 3 × 6, 3 × 8, 3 × 10, 3 × 12.
4 × 4, 4 × 6, 4 × 8, 4 × 10, 4 × 12.
6 × 6, 6 × 8, 6 × 10, 6 × 12.
8 × 8, 8 × 10, 8 × 12.

12 ft. to 22 ft. are ordinary lengths.
23 ft. to 26 ft. are less common.
27 ft. to 32 ft. are obtained with difficulty.

Yellow Pine: Sizes about the same as for spruce, also

2 × 14, 2 × 16.
3 × 14, 3 × 16.
4 × 14, 4 × 16.
6 × 14, 6 × 16.
8 × 14, 8 × 16.
10 × 14, 10 × 16.
12 × 14, 12 × 16.
14 × 14, 14 × 16.
16 × 16.

Lengths of yellow pine sticks are longer than for spruce and run up to 40 ft., and it is usually possible to obtain even 50 ft. lengths except for the largest sizes.

The cost of wooden beams depends upon the price of lumber per board foot. This is subject to considerable variation, and if a close estimate is desired, a dealer should be consulted.

56. Steel Beams. Such beams are usually made with a cross-section of the shape of the letter I in order to obtain a large moment of inertia from a comparatively small amount of material. They are rolled from solid pieces of steel in varying heights and thicknesses. In selecting such beams the manufacturer's handbooks should be consulted, and sections marked "standard" chosen since the selection of a "special" section is likely to cause delay in filling the order. These handbooks give all the properties of the beams, such as area, weight, moment of inertia, etc., and may be relied upon as accurate.

The cost of steel beams is dependent upon the weight of the beam, and upon the amount of punching, riveting and other work which has to be done. The price is usually figured on a "cent per pound" basis, the price of the unfabricated beam being taken as the base price, and the other prices added thereto. Other things being equal, the lightest beam having the requisite strength and stiffness is most economical. The base price is published from time to time in such engineering papers as the *Engineering News*, *Iron Age*, etc., e.g., in the *Engineering News* of Dec. 2, 1909, the f.o.b. Pittsburg price was quoted as 1.55 cents per pound for 3 inch to 15 inch I-beams and channels and 1.60 cents for depth greater than 15 inches. The freight rates for carload lots from Pittsburg as quoted in this same issue were as follows: To New York 16 cents, and to Boston 18 cents per 100 lbs. This price is for beams cut to length with a variation not to exceed $\frac{3}{8}$ in. more or less than specified. For cutting to more exact length and for other work the following schedule adopted in January 1902 gives the extra cost in cents.

1. For cutting to length with less variation than plus or minus $\frac{3}{8}$	0.15 per 100 lbs.
2. Plain punching one size hole in web only	0.15 " 100 "
3. Plain punching one size hole in one or both flanges	0.15 " 100 "
4. Plain punching one size hole in either web and one flange or web and both flanges	0.25 " 100 "
5. Plain punching each additional size hole in either web or flange, web and one flange, or web and both flanges	0.15 " 100 "

57. Examples of Beam Design.

Problem. Design wooden and steel beams for 12-ft. span. Beams to be supported at ends and to be loaded with a total uniform load (live and dead) of 1000 lbs. per foot. Allowable unit stresses to be those given in Art. 18 for steel and yellow pine. Fifty per cent of total load to be added in the case of the steel beam to allow for impact.

Solution. Maximum moment is at centre of beam and equals

$$\frac{1}{8}\ 1000\times 12\times 12 = 18{,}000 \text{ ft.-lbs.}$$

Maximum shear is at end of beam and equals 6000 lbs.

For wooden beam $M=\frac{1}{6}sbh^2$, $\therefore$ $18{,}000\times 12=\frac{1}{6}\ 1300bh^2$.

$$\therefore\ bh^2=997.$$

Either an 8×12-inch, 6×14-inch, or 4×16-inch beam has a value of bh^2 greater than that required and may be used.

The area of cross-section needed to carry shear may be determined by Eq. (16) and is given by the following expression:

$$A=\frac{18{,}000}{240}=75 \text{ sq.ins.}$$

Evidently the 4×16-inch beam is too small, and one of the other beams should be selected. The 6×14-inch is the cheapest and should be chosen if conditions permit. The longer side should always be placed parallel to the plane of the loads, i.e., vertical if the loads are vertical. This was the position assumed in solving for bh^2, and none of the beams selected would be strong enough if not so placed.

The bearing area on the abutment should also be determined. If the reactions were uniformly distributed over the bearing surface there would be needed $\frac{6000}{260}=23$ sq.ins. To allow for unequal distribution 50 per cent will be added to this, giving 34.5 sq.ins. The 6×14-inch beam would therefore need to extend $\frac{34.5 \text{ ins.}}{6}$ or, say, 6 ins. over the abutment.

For the steel beam the moment after allowance for impact is made $=27{,}000$ ft-.lbs.

$$\therefore\ \frac{I}{c}=\frac{27{,}000\times 12}{16{,}000}=20.25.$$

The term $\frac{I}{c}$ is known as the section modulus. Values of this for various beams are given in the handbooks issued by steel makers, and the lightest beam having a modulus equal to or greater than the above figure should be selected. A 10-inch I-beam, weighing 25 lbs. per foot has sufficient strength and will be chosen. A 9-inch, 25-lb. beam is also strong enough, but as this is just as expensive as the 10-inch beam

and not so strong, the 10-inch beam should be selected, provided conditions do not require the use of a shallower beam.

The distribution of shear over an I-beam is more complicated than in a rectangular beam. It will, however, be shown later that the shear is practically all carried by the web over which it is distributed almost uniformly. Making this assumption the area required in the web would be $\frac{9,000}{12,000}$ of a square inch, and as the actual area in the beam selected is far in excess of this the beam is evidently strong enough to carry the shear and hence may be used with safety.

The sizes selected were based upon the assumption that the beam would have no rivet or bolt holes and no other reductions in the cross-section area. If such reductions occur the value of I should be corrected to allow for the reduction in section, and the value of c also changed if the position of the neutral axis be shifted by the change in area. Methods of making such corrections will be given in full in Chapter V.

Another important element to be considered in selecting beams is that of vertical and horizontal stiffness. This will be considered in Art. 59, it being assumed for the present that the beams designed in this article are supported laterally where necessary, and that their vertical deflection is not excessive.

Problem. Design wooden and steel beams for a single track electric railway bridge of 12-ft. span carrying the electric car shown in Fig. 76.

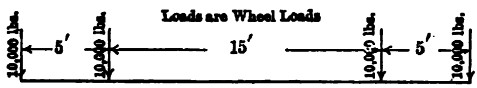


FIG. 76.

Assume track (ties, rails, etc.) to weigh 400 lbs. per lineal foot (200 lbs. per foot per rail), and each beam to weigh 40 lbs. per lineal foot. Allowance for impact to be 25 per cent. Unit stresses as in Art. 18.

Solution:

MAXIMUM MOMENT		MAXIMUM SHEAR	
Dead $=\frac{1}{8}240\times 12\times 12$	$=$ 4,300 ft.-lbs.	Dead $240\times 6=$	1,440 lbs.
Live $=20,000\frac{(6-1.25)^2}{12}$	$=$ 37,600 "	Live $10,000\times\frac{19}{12}=$	15,830 lbs.
Impact $=$	9,400 "	Impact $=$	3,960 "
Total moment $=$	51,300 "	Total shear $=$	21,230 "

Steel Beam. For steel beam assuming no reduction due to bolt holes

$$\frac{I}{c}=\frac{51,300\times 12}{16,000}=38.5.$$

$$\text{Web area needed for shear}=\frac{21,230}{12,000}=1.77 \text{ sq.ins.}$$

A 12-in. I-40 lbs. is large enough for bending and as it has a web area of 5.52 sq.ins. its strength in shear is far greater than necessary.

As the actual weight of the beam equals that originally assumed no recomputation is necessary. A considerable error might, however, have been made in the original assumption without requiring a recomputation since the moment and shear due to the weight of beam is a very small percentage of the total moment and shear.

Wooden Beam. For wooden beam, neglecting impact,

$$41{,}900\times 12=\frac{1}{6}\,1300bh^2. \quad \therefore \quad bh^2=2310.$$

$$\text{Area needed for shear}=\frac{3}{2}\,\frac{17{,}270}{120}=216 \text{ sq.ins.}$$

One beam 14×16 ins. with 16 ins. side vertical fulfils both requirements and will be chosen. Its weight is somewhat in excess of that assumed but as its strength is also in excess of the requirements no revision need be made.

58. Composite Beams. The cases just treated are of simple beams only, but it sometimes happens that composite beams are used, as for example a so-called flitch-beam consisting of two wooden beams and a steel plate bolted together and used as one beam. Another example is that of two beams of unequal size laid side by side. For both of these cases the load carried by each member is in proportion to the product of its moment of inertia and modulus of elasticity and can be easily computed. Still another case is that of one beam laid on the top of another, but not riveted to it. Such a beam is of slightly greater strength than two beams laid side by side; the additionl strength is due, however, to friction between the beams and should be neglected in design. If the beams are riveted together with a sufficient number of rivets to carry the longitudinal shear which would exist at the plane of contact, assuming the beam to be solid, they may be figured as one beam with a cross-section corresponding to that of the combination. Reinforced-concrete beams form the most important class of composite beams, but these will not be considered in this book. For a full discussion of these beams the student is referred to the valuable and thorough treatment of such beams in either "Concrete, Plain and Reinforced," by Taylor and Thompson, or "Principles of Reinforced Concrete Construction," by Turneaure and Maurer.

59. Stiffness. Beams are seldom used for bridge spans exceeding 30 ft. in length, since above that span the ratio of length to

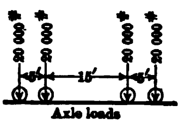
depth is so great that the deepest beam made, a 30-inch I-beam, lacks sufficient stiffness. It is common to specify that bridge members exposed to bending shall where possible have a depth not less than one-twelfth the span. For buildings longer beams are admissible, but the length should generally be restricted to twenty times the depth for floors and less than that if the floor be subjected to vibration and shock. In roofs somewhat longer beams may be used.

When the beams are not supported laterally, the ratio of length to width of the compression flange should be considered and the allowable unit stresses reduced accordingly. A rule sometimes adopted is to allow 16,000 lbs. for ratios of $\frac{L}{b} \not> 20$ and to reduce this uniformly to 8000 lbs. for $\frac{L}{b}=70$. L = unsupported length and b = width of flange. For spans of considerable length it is usually more economical to use lateral bracing between the top flanges of the beams than to use the heavier beams that would otherwise be necessary. It is assumed in the problem of Art. 57 that this has been done.

PROBLEMS

37. *a.* Design a steel I-beam stringer for bridge and loads given in Probs. 34 and 35, using unit values of Art. 18, and allowing 75 per cent for impact.

b. Design a yellow-pine stringer for the same bridge, using unit values of Art. 18 and neglecting impact. Proportion stringer for bending only, but compute the maximum intensity of longitudinal shear in the stringer selected.



PROB. 38.

PROB. 39.

38. Design a steel I-beam stringer for an electric railway bridge. Bridge to be a single track bridge with 15 ft. panels and stringers located symmetrically with respect to the rails. Assume total dead weight of track and stringers to be 600 lbs. per lineal foot of bridge. Use live loads shown and allow 25 per cent for impact. Unit stresses as given in Art. 18.

39. Design steel I-beam stringers for the bridge of Prob. 38 for the electric locomotive shown. All other conditions to be the same as in the previous problem.

CHAPTER V

PLATE GIRDER DESIGN

60. Plate Girders Defined. A plate girder is essentially an I-beam made, not out of one solid piece of metal, but out of a number of pieces riveted together. Fig. 3 shows a plate girder bridge, and Fig. 77 shows the cross-section of a typical plate girder.

Plate girders are rarely made of greater depth than 10 ft. 6 ins.

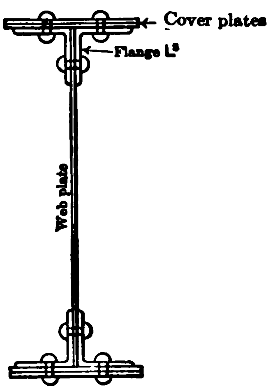


FIG. 77.—Cross-section of a Plate Girder.

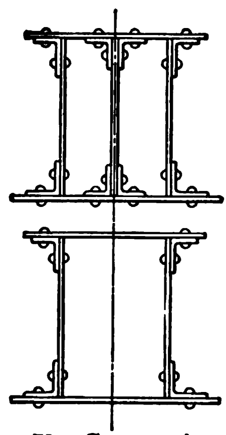


FIG. 78.—Cross-sections of Two Box Girders.

owing to difficulties in transportation by rail, and a length of 100 ft. is seldom exceeded for the same reason although girders of 125 ft. in length have been made and shipped in one piece. Occasionally plate girders are made in sections and spliced in the field, but this expedient is not common and should not be adopted except to meet some unusual condition.

A plate girder with more than one web is called a box girder. It is used in situations where great strength with limited depth is required.

61. Plate Girder Web Theory. Plate girder webs may be pro-

portioned on the assumption that all the transverse shear is uniformly distributed over the net area of the web, and that this may be taken as three-fourths the gross area without material error. This may be expressed by the following formula in which A = gross area of web, V = total external shear, and v = allowable intensity of internal shear.

$$A = \frac{4V}{3v} \qquad v = \frac{VQ}{bI} \qquad (16a)$$

Q = statical moment

That the above formula is essentially correct is shown by the following demonstration:

Let Fig. 80 represent the square prism, *abcd*, from the web of the plate girder shown in Fig. 79 and let it be assumed that

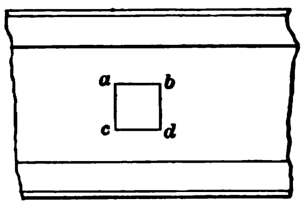


FIG. 79.

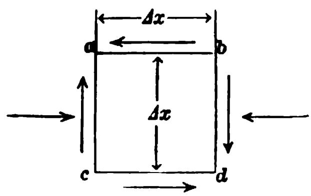


FIG. 80.

there are shearing forces acting on all four surfaces, and direct stresses on the two vertical surfaces.

Let t = its thickness = thickness of web.

s_h = intensity of the shearing force on the surface *ab*.

s_h' = intensity of the shearing force on surface *cd*.

s_v = intensity of shearing force on surface *bd*.

s_v' = intensity of shearing force on surface *ac*.

f = intensity of the direct stress on surface *bd* (assumed compression for convenience).

f' = intensity of the direct stress on surface *ac* (also compression).

Application of the equations of equilibrium give the following results:

$$s_v t\Delta x = s_v' t\Delta x. \qquad \therefore \quad s_v = s_v'.$$

$$s_h t\Delta x + f t\Delta x = s_h' t\Delta x + f' t\Delta x. \qquad \therefore \quad s_h = s_h' + (f' - f).$$

$$(s_h t\Delta x)\Delta x - (s_v t\Delta x)\Delta x + [(f - f')t\Delta x]\frac{\Delta x}{2} = 0. \qquad \therefore \quad s_h = s_v + \frac{(f' - f)}{2}.$$

As the distance Δx becomes infinitesimal $\frac{f' - f}{2}$ approaches zero, hence at the limit $s_h = s_h' = s_v$.

It therefore follows that the intensity of the horizontal shear in the web of a plate girder at any point on a vertical plane equals the intensity at the same point on a horizontal plane.

Since the intensities of the vertical and horizontal shears are equal, it is evident that the distribution of the vertical shear can be determined by the application of formula (13), from which it at once follows that the vertical shear is distributed over the web with approximate uniformity since Q is the only term in the equation affected by the distance from the neutral axis, and its value changes much more slowly than does the distance from the axis. The numerical examples given in Art. 64 show the degree of approximation of this assumption for certain typical girders.

In many girders the thickness of the web is determined by imposing restrictions upon its minimum thickness to prevent undue corrosion. For railroad bridges it is common to specify that the web shall be not less than three-eighths inch thick.

62. Plate Girder Flanges. Theory. Formula (14) applies to girders as well as beams. It is, however, in inconvenient form for use and may be replaced in practice for symmetrical plate girders by a less accurate but more easily applied formula. The formula recommended for plate girder flanges is as follows:

$$A = \frac{M}{sh} - \frac{1}{12}th_1{}^{\dagger} \qquad (17)$$

The derivation of this formula is as follows:

Let A = net area in square inches of tension flange (through rivet holes).

h = distance in inches between centres of gravity of the two flanges.

s = allowable unit stress in bending.

t = thickness of web in inches.

h_1 = depth of web in inches.

M = maximum bending moment on given section in inch-pounds.

I = total moment of inertia of gross cross-section about neutral axis.

A_1 = gross area of each flange.

---

† These formulas should not be used for unsymmetrical girders nor for very shallow girders with heavy flanges, where the distance between centres of gravity of flanges is much less than the total depth of the girder, nor for other abnormal cases.

h_2 = depth out-to-out of flanges.

$I_{c.g.}$ = moment of inertia of each flange about its own centre of gravity.

The following equation for I may now be written:

$$I = \cancel{2I_{c.g.}} + 2A_1\left(\frac{h}{2}\right)^2 + \frac{1}{12}th_1^3.$$

The term $2I_{c.g.}$ is small in comparison with the other terms and may be omitted without serious error, this being on the safe side. In consequence the value of $\frac{I}{c}$ for a symmetrical cross-section may be written thus:

$$\frac{I}{c} = \frac{\frac{2A_1h^2}{4} + \frac{1}{12}th_1^3}{\frac{h_2}{2}} = \frac{A_1h^2}{h_2} + \frac{1}{6}\,\frac{th_1^3}{h_2}.$$

But $$\frac{I}{c} = \frac{M}{s}. \quad \therefore \quad \frac{M}{s} = \frac{A_1h^2}{h_2} + \frac{1}{6}\,\frac{th_1^3}{h_2};$$

hence, $$A_1 = \frac{M}{s}\,\frac{h_2}{h^2} - \frac{1}{6}\,\frac{th_1^3}{h_2}\,\frac{h_2}{h^2} = \frac{M}{s}\,\frac{h_2}{h^2} - \frac{1}{6}\,\frac{th_1^3}{h^2}$$

$$A_1 = \frac{M}{sh}\left(\frac{h_2}{h}\right) - \frac{1}{6}th_1\left(\frac{h_1}{h}\right)^2.$$

For girders with ordinary depths (say $\frac{1}{8}$ to $\frac{1}{12}$ the span) the value of h is seldom larger than that of h_1 while it is usually smaller.[1] If $\frac{h_1}{h}$ be therefore assumed as unity the last term in above equation will ordinarily be less than its true value, and since this term is small compared with the term involving M, the slight change in its value by the above approximation will affect the value of A but little, and that on the safe side, since it reduces the value of the negative term. This approximation will therefore be made.

The assumption that $\frac{h_2}{h}$ = unity will also be made. This is on the unsafe side, since h is always less than h_2, and to assume

[1] Most specifications forbid the use in design of a value for h greater than h_1 even if it actually exists. It is good practice to proportion girders so that such a condition will not occur.

it equal gives a smaller value for A_1 than is required. The error in making this assumption is largest in shallow girders having large flanges as may be seen in the numerical examples given later.

By making the above approximation the formula becomes

$$A_1 = \frac{M}{sh} - \frac{1}{6}\,th_1.$$

For material in compression it is customary to make no deduction whatever for rivet holes since it is assumed that the rivet which is driven while hot and ordinarily under high pressure fills the hole so completely as to become an integral portion of the material. This is open to some doubt in the case of thick material or hand-driven rivets, and may be vitiated at any section by a loose rivet, but for most cases this assumption is probably a reasonable one. For sections in tension full allowance for rivet holes must be made, since under no circumstances can tension be transmitted through a rivet hole.

The last term in formula (17) represents the bending resistance of the web. As there are usually vertical rows of rivets in the web for floor-beam connections, stiffener angles, etc., and as these may occur at the section carrying maximum moment they must be considered.

To allow for such holes, it may be assumed that a vertical row of holes one inch in diameter and $2\frac{3}{4}$ inches apart may occur in the tension-half of the web. This would decrease the moment of inertia by $\frac{2}{11}$ approximately, thus making the last term in the equation $\frac{9}{66}\,th_1$ or say $\frac{1}{8}\,th_1$.

Allowance for rivet holes in the tension flange must also be made. This may be done by substituting A for A_1, which is in reality equivalent to providing for rivet holes in both flanges. This may seem excessive, but some excess is necessary since the section has been considered as solid and with its neutral axis at mid-height, whereas in reality the influence of the rivet holes in the tension portion is to shift the neutral axis from the centre, thus diminishing the moment of inertia and increasing the distance from neutral axis to extreme fibre. The substitution of A for A_1 is, however, more than sufficient for this purpose and helps to diminish the error made in placing $\frac{h_2}{h}$ = unity.

This modification gives the following formula, which is adopted by many engineers:

$$A = \frac{M}{sh} - \frac{1}{8} th_1.$$

The last term in this equation represents the resistance of the web to bending. Owing to the difficulty in satisfactorily splicing the web many engineers disregard its resistance to bending and use the formula

$$A = \frac{M}{sh}.$$

It is believed, however, that ample provision has been made in formula (17) for insufficient web splices in long girders by putting the term for web resistance as $\frac{1}{12} th_1$.

The assumptions made in deriving formula (17) are of such a character as to make the formula inaccurate for girders having great depth in proportion to their length. Such girders are not common in bridges but are sometimes used in architectural work, and should be solved by the direct application of formula (14).

It should be stated, furthermore, that experimental knowledge of the distribution of stress in plate girders is insufficient to permit a confirmation of the accuracy of formulas of the type of (17). Formulas of this character have, however, been in use for many years with satisfactory results, and may well be considered as safe working formulas. Formula (17) is more conservative than that usually employed.

63. Degree of Approximation of Flange Formula. In order to show the degree of approximation of formula (17), in comparison with formula (14), the problems which follow have been inserted.

Problem. Compute allowable bending moment, M, for the girder shown in Fig. 81. Assume no intermediate web stiffeners, and hence only one rivet hole (flange rivet) in tension half of girder. Allowable unit stress $= s$.

Allowable moment by beam formula.

	Area in sq. ins.
Top angles, 2–6″×4″×½″, at 4.75,	9.5 gross
Bottom angles, 2–6″×4″×½″, at (4.75 − 0.5) =	8.5 net
Web, 29″×½″. =	14.5 net
Total effective area of cross-section =	32.5 sq. ins.

Distance of centre of gravity of cross-section above axis xy

$$=\frac{1\times1\frac{1}{2}\times13\frac{1}{8}}{32.5}=0.6 \text{ in.}$$

$$h=30.25 \text{ ins.}-3.98 \text{ ins.}=26.27 \text{ ins.}$$

Let Ixy = moment of inertia of *gross* section about axis xy and $I_{c.g.}$ = moment of inertia of any piece about an axis parallel to xy and passing through the centre of gravity of the piece.

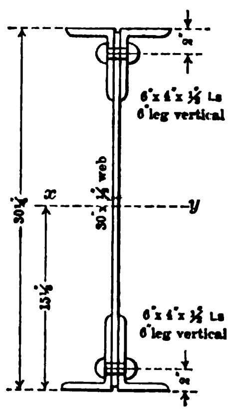


FIG. 81.

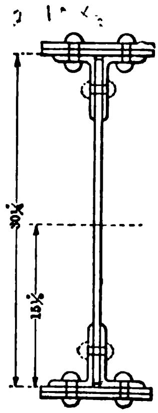


FIG. 82.

I_{xy} of webs $=\frac{1}{12}\cdot\frac{1}{2}\cdot30\cdot30\cdot30$	$=1125$
I_{xy} of angles $=I_{c.g.}$ of angles $+\frac{A_1h^2}{2}=4\times17.4+4\times4.75\times(13.13)^2$	$=3345$
Total I_{xy},	$=4470$
Deduct for flange rivet hole $1\frac{1}{2}\times1\times(13.13)^2$	$=259$
I of net section about axis xy	$=4211$
To obtain I for net section about axis passing through centre of gravity deduct, 32.5×0.6^2	$=12$
I of net section about neutral axis	$=4199$

$$\frac{I}{c}=\frac{4199}{(15\frac{1}{8}+0.6)\text{ ins.}}=267.$$

$$\therefore\quad M=s\frac{I}{c}=267s.$$

Allowable Moment by Formula (17). By transformation of terms in the formula, M is found to be given by the expression

$$M=(A+\tfrac{1}{12}th_1)sh=(8.5+1.25)(s)(26.27)=256s.$$

Since the allowable moment as computed by formula (17) is less than that given by the beam formula, the formula for this girder is on the safe side. The approximation is about 4 per cent.

Problem. Compute allowable bending moment for the same girder, assuming a row of rivet holes in the web at the point of maximum bending moment in addition to the flange rivet holes.

Solution. For practical reasons the web rivets nearest the flanges should be located not less than $1\frac{1}{2}$ ins. from the edge of the flange angles, or for this girder $7\frac{1}{2}$ ins. from the backs of the angles. In order to get even rivet spacing let this distance be made $7\frac{5}{8}$ ins., thus giving 15 ins. for the distance between these rivets and permitting the use of five spaces at 3 ins. for the horizontal rivets between flanges.

Assume that only the rivet holes in that portion of girder below the neutral axis, i.e., the tension half, need be deducted.

Net area of cross-section $= 32.5 - 3\times\frac{1}{2} = 31.0$.

The position of the centre of gravity of the crosss-section above the axis $xy = \dfrac{1\frac{1}{2}\times 13\frac{1}{8} + \frac{1}{2}(7\frac{1}{2}+4\frac{1}{2}+1\frac{1}{2})}{31.0} = 0.85$ in.

The deduction from I_{xy} to allow for rivet holes will be

$$1\times 1\frac{1}{2}\times(13.13)^2 + \frac{1}{2}\times(7.5^2+4.5^2+1.5^2) = 298.$$

$$\therefore\ I \text{ of net section about axis } xy = 4470 - 298 = 4172$$
$$\text{and about neutral axis} = 4172 - 31.0\times 0.85^2 \quad = 4150$$

$$\therefore\ \frac{I}{c} = \frac{4150}{15.12+0.85} = 260,$$

and
$$M = \frac{sI}{c} = 260s.$$

The approximation in this case is somewhat less than 2 per cent and is also on the safe side.

Problem. Compute allowable bending moment for the same girder assuming 2–10×½ in. plates to be added to each flange and no web rivets at the critical section, see Fig. 82. (Note that the horizontal and vertical flange rivets are frequently staggered and hence a section containing the vertical rivets may not contain horizontal rivets; also that the material in the horizontal legs of the flange is thicker than that in the vertical leg, hence the reduction due to rivet holes is larger.)

Allowable Moment by Beam Formula.

	Area in sq.ins.
Top angles, 2–6″×4″×½″ at 4.75	= 9.5 gross
Bottom angles, 2–6″×4″×½″ at (4.75 − 0.5)	= 8.5 net
Top plates, 2–10″×½″ at 5	= 10.0 gross
Bottom plates, 2–10″×½″ at (5 − 1)	= 8.0 net
Web 30″×½″	= 15.0 gross

Total effective area of cross-section = 51.0 sq.ins.

Computation of h.

Gross area of one flange = 19.5 sq.ins.

From back of angles to centre of gravity of angles = 1.99″

Hence from centre of gravity of angles to centre of gravity of flange

$$=\frac{10\times(1.99+0.5)}{19.5}=1.27''.$$

$$\therefore\ h=30.25''-2\times(1.99''-1.27'')$$
$$=28.81''.$$

Allowance for rivet holes $=1\times1\frac{1}{2}\times2=3$ sq.ins, hence centre of gravity of cross-section above axis xy

$$=\frac{3\times(15\frac{1}{8}+\frac{1}{2})}{51}=0.9''.$$

Allowable Moment by Beam Formula.

I_{xy} of web		$=1125$
I_{xy} of angles		$=3345$
I_{xy} of plates $=20\times(15\frac{5}{8})^2$		$=4883$
Total I of gross section about axis xy		$=9353$
Deduct for rivet holes $3\times(15\frac{3}{8})^2$	$=709$	
Correction for I about neutral axis $=51\times0.9^2=$	41	
		750
I of net section about neutral axis		$=8603$

$$\frac{I}{c}=\frac{8603}{16\frac{1}{8}+0.9}=505.$$

$$\therefore\ M=505s.$$

Allowable Moment by Formula (17)

$$M=(A+\tfrac{1}{12}th_1)sh=(8.5+8.0+1.25)(s)(28.81)=511.4s.$$

Here the formula errs on the unsafe side, the approximation being slightly over one per cent.

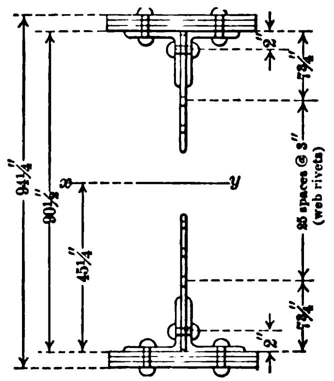


FIG. 83.

Problem. Compute allowable bending moment for girder shown in Fig. 83. Assume a vertical row of rivets, 3 ins. centre to centre, at the point of maximum moment, and that all flange rivets shown occur in the same cross-section as the web rivets.

$$\text{Composition of girder}\begin{cases}1\text{ web }90''\times\frac{1}{2}''.\\ 2\text{ top angles }6''\times6''\times\frac{3}{4}''.\\ 3\text{ top plates }16''\times\frac{5}{8}''.\\ 2\text{ bottom angles }6''\times6''\times\frac{3}{4}''.\\ 3\text{ bottom plates }16''\times\frac{5}{8}''.\end{cases}$$

Allowable Moment by Beam Formula.

	Area in sq.ins.
Top angles, 2–6″×6″×¾″, at 8.44	= 16.88 gross
Bottom angles, 2–6″×6″×¾″, at (8.44 − 2×¾)	= 13.88 net
Top plates, 3–16″×⅝″, at 10	= 30.00 gross
Bottom plates, 3–16″×⅝″, at (10 − 1.25)	= 26.25 net
Web 90″×½″ − 14″×½″ (Deducting rivet holes in lower half)	= 38.00 net
Total effective area of cross-section	= 125.01 sq.ins.

Computation of h.

Back of angles to centre of gravity of angles $=1.78$ ins.

$$\frac{\text{Moment of plates about c.g. of angles}}{\text{Gross area of flange}}=\frac{30\times2.72}{46.88}=\frac{1.74}{0.04}\text{ ins.}$$

$$\therefore\quad h=90.5''-0.08''=90.42''.$$

Moment about xy of rivet holes in tension half of girder is as follows:

Web	$\frac{1}{2}(43.25+37.5+34.5+\ldots+1.5)=\frac{1}{2}\times296.8$	$=148.4$
Flange	$1\frac{1}{2}\times43\frac{1}{4}+5.25\ (45.81)$	$=305.4$
	Total	$=453.8$

Distance of neutral axis above $xy=\dfrac{453.8}{125.0}=3.63''$.

Maximum value of c $=50.75''$.

I_{xy} for web	= 30,375 gross
I_{xy} for angles $=112+2\times16.88\times(43.47)^2$	= 63,906 gross
I_{xy} for plates $=17+2\times30\times(46.187)^2$	= 128,014 gross
I of gross section about axis xy	= 222,295
Deduct for rivet holes, $\frac{1}{2}[(43.25)^2+(37.5)^2+\ldots(1.5)^2]+1\frac{1}{2}(43.25)^2+5.25(45.81)^2$	= 18,050
I of net section about axis xy	= 204,245
Correction for I about neutral axis	= 1,647
I of net section about neutral axis	= 202,598

$$\frac{I}{c}=\frac{202{,}598}{50.75}=3992\ \therefore\ M=3992s$$

Allowable Moment by Formula (17)

$$M=(A+\tfrac{1}{8}th_1)sh=(13.88+26.25+3.75)90.42s=3967s.$$

In this case the approximation equals about one-half per cent on the safe side.

The examples that have been given show that formula (17) gives for ordinary girders a very close approximation to the value obtained by the ordinary beam formula $M=\frac{sI}{c}$. It is much more convenient to use, since by it the required flange area can be directly computed, after the web is determined, by estimating the value h, which can be done by the experienced computor with little error. The actual application of the formula to the design of a girder will be shown later.

64. Degree of Approximation of Shear Formula. To show the degree of approximation involved in the ordinary assumption that the vertical shear is distributed uniformly over the net area of web, the maximum intensity of shear will be computed for the girders in the last article. This occurs at the neutral axis and will be computed in each case, using for Q the statical moment of the portion of the gross area of the girder above this axis, although the same numerical result would be obtained by considering the portion below the axis since the statical moment of the entire section about the neutral axis equals zero.

Problem. Compute maximum intensity of shear in the web of the girder shown in Fig. 81, assuming a vertical row of web rivets to occur at section of maximum shear.

Solution. Q for angles $=9.5\times(15.12-2.84) \qquad =116.7$

Q for web $=\frac{1}{2}\times(15.00-0.85)\times\frac{(15.00-0.85)}{2}= \quad 50.0$

Total value of $Q \qquad =166.7$

From formula (13) $\quad v=V\frac{166.7}{4150\times\frac{1}{2}}=.0803V.$

(See page 115 for value of I.)

By the assumption that the shear is uniformly distributed over three-fourths the gross area of the web the following result is obtained:

$$v=\frac{V}{\frac{3}{4}\times 15}=0.0888V.$$

The value thus obtained is on the safe side by about ten per cent.

Problem. Compute maximum intensity of shear in the web of the girder shown in Fig. 83, assuming that no cover plates occur at section of maximum shear.

Solution. Net area $= 68.76$[1] sq.in.

Centre of gravity above $xy = \dfrac{148.4+1.5(43.25+44.875)}{68.76} = 4.08''$

I_{xy} for gross area of web and angles $=94,281$

Deduct for rivets $\frac{1}{2}[(43.75)^2+(37.5)^2+-(1.5)^2]$
$+1\frac{1}{2}[(43.25)^2+(44.875)^2]$ $=10,052$

I of net section about xy $\quad 84,229$

Correction for I about neutral axis $=68.76\times 4.08^2 = 1,144$

I of net section about neutral axis $=83,085$

Q of angles $=16.88\times 39.39 = 665$

Q of web $=(45-4.08)(\frac{1}{2})\dfrac{(45-4.08)}{2} = 419$

$\quad 1084$

$$\therefore \quad v=\frac{1084V}{83,085\times\frac{1}{2}}=0.0261V.$$

By the assumption that the shear is distributed over three-fourths the gross area of web, the following value would be obtained:

$$v=\frac{V}{33.75}=0.0296V.$$

This result is again on the safe side by about ten per cent.

If in either of the cases just considered the shear had been assumed as distributed uniformly over the gross area, a considerable error on the unsafe side would have resulted.

Although the two girders considered do not represent extreme cases, it is believed that the results are representative, and that for all ordinary cases the assumption that the shear is distributed uniformly over three-fourths the gross area of the web is a safe and reasonable working hypothesis.

It should be said that the shear may be assumed as distributed uniformly over the gross area of the web if the allowable shearing stress be modified accordingly, but the fact that rivets may perhaps fill their holes so perfectly that they may be considered to transmit shear equally as well as compression should not be regarded as a reason for making such an assumption, since it is the influence of the flange which is really the important factor in determining the distribution of the shear.

---

[1] In determining the area it is assumed that the maximum shear may occur at the point where the first cover plate begins, hence vertical flange rivets may occur at the section but cover plate should be ignored.

For I-beams the shear may be treated in somewhat the same manner. The values obtained for a 10-in. I, 25 lbs. per foot, are as follows, assuming no rivet holes in cross-section:

$$\text{By formula (13),} \qquad v=\frac{14.02}{0.31\times 122.1}V=0.370V,$$

$$\text{By common assumption, } v=\frac{V}{2.325}=0.430V.$$

The error in this case is on the safe side, but had the gross web area been used the error would have been on the *unsafe* side.

For a 24-in. I, 80 lbs. per foot, the corresponding values would be $.097V$ and $.111V$ the error in this case being also on the safe side.

As it is seldom that the shear in I-beams is a controlling factor in the design this approximation is not of great importance. For cases where the shear controls, a liberal allowance should be made in determining the web thickness, or else the actual stress should be determined by the more exact formula.

65. Allowance for Rivet Holes. In the design of girders it is necessary to make due allowance for the tension-flange rivet holes in advance of the completion of the detailed drawings. No accurate rule for doing this can be given since the actual reduction of strength by rivet holes needs more thorough experimental investigation than it has yet received. The following rules may, however, be used as a guide:

1. For flanges with cover plates, and angles with legs wider than 4 in., assume that both vertical and horizontal rivet holes may occur in the same section, these holes being as shown in Fig. 84.

2. For flanges with flange angles of 4 in. or less in width, and with cover plates, deduct two holes from each section as shown in Fig. 85.

3. For flanges without cover plates deduct one hole from each section as shown in Fig. 86.

In all cases the rivet hole should be assumed to have a diameter $\frac{1}{8}$ in. more than the nominal diameter of the rivet. This is necessary since the hole is usually punched with a diameter $\frac{1}{16}$ in. greater than that of the cold rivet, and the edges of the hole may be damaged somewhat in punching. The rivets commonly used in structural work are $\frac{7}{8}$ in. diameter, hence, for

these rivets the hole should be assumed as one inch in diameter. In light work, $\frac{3}{4}$ in. or $\frac{5}{8}$ in. rivets are occasionally used and in very heavy girders one inch rivets are sometimes employed.

It should be stated that while it is seldom that more than three rivet holes actually occur in the same section of the tension flange, the fact that a zig-zag section through holes not in the same right section may have a less net area than that in any

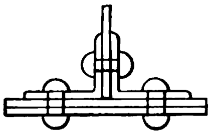


FIG. 84.

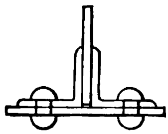


FIG. 85.

given right section must be considered. For example, the zig-zag section AB in Fig. 87 may have a much smaller net area than the cross-section AC, and if the distance BC is small may have a net area but little if any greater than that on a section like DE. If desirable the actual net area of a section like AB may be computed or determined graphically, although experimental results are lacking to show that the strength of the flange varies

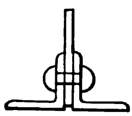


FIG. 86.

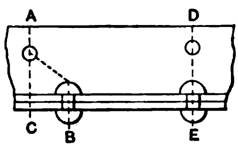


FIG. 87.

directly with such an area. It is, however, wise to make liberal allowance for rivet holes and if the pitch CB is less than $2\frac{1}{2}$ ins. to ordinarily allow for three holes in the section as shown in Fig. 84.

Attention should be called to the fact that while the maximum moment on a girder ordinarily occurs at only one section, and that at this section the rivet pitch may be a maximum, a maximum flange stress is developed wherever a cover plate ends, and the rivet pitch at such a point may be little if any larger than the minimum allowable pitch.

66. Example of Girder Design. An example of the complete design of the cross-section of a girder will now be given.

Problem. Determine cross-section of a girder to carry a maximum bending moment of 1,250,000 ft. lbs. and a maximum shear of 160,000 lbs. Depth of girder back to back of flange angles $= 48\frac{1}{2}$ ins.

Allowable unit stresses,

Bending	16,000 lbs. per sq.in.
Shear	12,000 "

Solution. *Web.* Net area of cross-section required $= \dfrac{160,000}{12,000} = 13.33$ sq.ins.

Depth of web 48 ins. Thickness of web $\dfrac{13.33}{\frac{3}{8}\times 48} = 0.37$ in. or $\frac{3}{8}$ in.

Flange. Assume h to equal depth of web $= 48$ ins.

Trial $A = \dfrac{1,250,000\times 12}{16,000\times 48} - \dfrac{1}{12}\cdot\frac{3}{8}\cdot 48 = 19.53 - 1.50 = 18.03$ sq.ins.

Trial section:

2 angles $6''\times 4''\times\frac{5}{8}''$ at 5.86	$= 11.72 - 2.50 =$	9.22 sq.ins.
2 plates $14''\times\frac{3}{8}''$ at 5.25	$= 10.50 - 1.50 =$	9.00 "
(Rivets in flange assumed as in Fig. 83)		18.22

Computation of h for this section.

Centre of gravity of angles from back of angles $= 1.03''$ (from handbook).

Correction to allow for cover plates $\dfrac{10.5\times 1.40}{10.5 + 11.72} = 0.66''$.

$$\therefore\quad h = 48.5'' - 2\times(1.03'' - 0.66'') = 47.8''.$$

Hence, original assumption for value of h, while slightly too large, is sufficiently accurate and the trial section may be used.

67. Rivets and Riveted Joints. The flange rivets form the only connection between the flanges and the web, hence the determination of the proper size and distance apart of these rivets is an essential feature in girder design. The diameter of the rivet is ordinarily fixed by practical considerations, the common practice for structural work being to use $\frac{7}{8}$ in. rivets. The distance apart of the rivets has to be computed. The question of riveted connection between two members is also of great importance.

Thorough treatment of rivet spacing is found in treatises upon mechanics and will not be given here. The essential points with which the structural designer must be thoroughly conversant, are as follows.

A riveted connection may fail in one of the following ways:

a. By the shearing of the rivets.

b. By the crushing of the rivets or of one of the pieces upon which they bear.

c. By the tearing of the rivets through one of the connected pieces.

Under *a* it should be noted that the allowable shearing value of the rivet may be found by multiplying its cross-section area by the allowable shearing stress per square inch, and that the area of a $\frac{7}{8}$ in. rivet is 0.60 sq.in., and of a $\frac{3}{4}$ in. rivet 0.44 sq.in.

In designing rivets to resist shear the plane upon which the maximum shear occurs must always be determined. If the maximum shear be equally distributed over two planes the rivet is said to be in *double shear*.

The permissible bearing, or crushing, strength of a rivet against a given plate is determined by multiplying the allowable bearing strength per square inch by the diameter of the rivet and the thickness of the plate in question.

To satisfy the requirements stated in *c*, use the following empirical rule: Rivets may not be spaced closer than three times the diameter, and the distance of a rivet from the edge or end of a piece may not be less than $1\frac{1}{4}$ in. for a $\frac{7}{8}$ in. rivet if the edge in question be rolled or planed, or $1\frac{1}{2}$ in. if it be sheared, though where possible this distance should be at least twice the diameter of the rivet. For other sizes of rivets proportional allowances should be made.

The two following examples show the application of these rules to some simple cases:

Problem. Determine number of $\frac{7}{8}$ in. rivets needed in row *a* to connect plates shown in Fig. 88.

Allowable shearing stress = 7,500 lbs. per sq.in.
Allowable bearing stress = 15,000 "

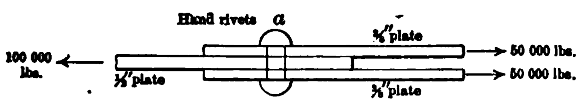


FIG. 88.

Solution. The maximum shear on a plane through the rivets = 50,000 lbs. As the rivets are $\frac{7}{8}$ in. diameter, one rivet will carry in shear $7500 \times 0.6 = 4500$ lbs., hence, if the strength of the joint is limited by the shearing strength of the rivets there are needed $\frac{50,000}{45} = 11(+)$, or 12 rivets. These rivets are limited in crushing strength by the $\frac{1}{2}$-in.

plate which carries 100,000 lbs. The value of the rivet in bearing against this plate equals $\frac{7}{8}\times\frac{1}{2}\times 15{,}000=6560$ lbs., and the number required $=\frac{100{,}000}{6560}=15(+)$, or 16. As this is larger than the number needed to prevent shearing 16 rivets must be used.

Problem. Determine number of $\frac{7}{8}$-in. rivets required to connect plates in joint shown by Fig. 89. Use same rivet values as in previous problem.

Solution. The maximum shear$=100{,}000$ lbs. and occurs between plates 2 and 3, or 4 and 5. The number of rivets needed to carry this shear$=\frac{100{,}000}{4500}=22(+)$ or, say, 23.

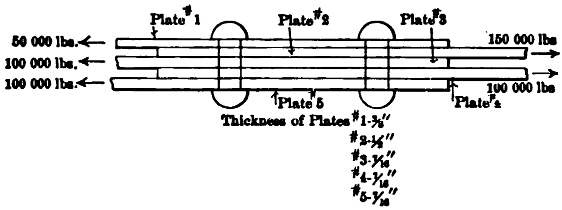


FIG. 89.

The bearing strength is evidently limited by plate No. 2, which carries 150,000 lbs. and has a thickness of $\frac{1}{2}$ in., this producing a greater stress on the rivets than would be the case for the $\frac{7}{16}$ in. plate carrying 100,000 lbs.

The number required for bearing$=\frac{150{,}000}{6560}=22(+)$, or 23, hence for this joint the number of rivets is limited by either shear or bearing.

The examples just given illustrate methods of computing connection rivets for plates carrying direct stress. Sometimes, however, it is necessary to transmit torsion as well as direct stress by means of rivets. Such a condition often occurs in steel-frame building construction where the connections of girders to columns must be given considerable rigidity to provide proper transmission of the wind stresses.

The condition commonly occurring in such a case is represented diagrammatically by Fig. 90, in which the load P is applied at a distance x from the centre of the group of rivets, thus producing a torsion Px which must be carried by the rivets in addition to the direct load P.

If the torsion be produced by a couple, as in Fig. 91, then the vertical load upon the rivets will be zero and a rivet may be legiti-

mately assumed to offer a resistance to torsion varying directly with its distance from the centre of gravity of the group of rivets, and acting at right angles to the line connecting it with the centre of gravity. Upon this basis the rivets at a, Fig. 91, should each be computed as equally stressed up to the allowable working load, while the other rivets would carry such proportion of the

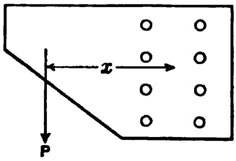


FIG. 90.

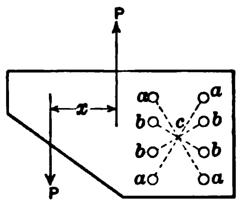


FIG. 91.

working stress as the distance cb is to ca when c is the centre of gravity of the group of rivets.

The resistance to torsion of such a group of rivets may therefore be expressed as follows:

Let r = the allowable working value of the most stressed rivet.

I = summation of the squares of the distances from the centre of gravity of the group of rivets to each rivet.

d = distance from centre of gravity of group of rivets to the most stressed rivet $= ac$, Fig. 91.

R = resistance to torsion of the group of rivets.

Then $$R = \frac{r}{d} I. \qquad (18)$$

For the case shown in Fig. 90 the above method must be modified to allow for the effect of the vertical load. To make this correction it is only necessary to determine the allowable resistance to torsion consistent with the rivet carrying its share of the vertical load.

This may best be done graphically by the method indicated in Fig. 92. In this case the total vertical load is 20,000 lbs., of which each rivet is assumed to carry 2500 lbs. With an allow-

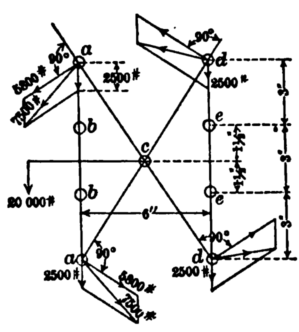


FIG. 92.

able total working value of say 7500 lbs. per rivet, the component at a perpendicular to the line ac is found graphically to be 5800 lbs. The corresponding allowable components of the stress in the rivets at d, b, and e are larger, hence the rivets at a furnish the minimum resistance to torsion for the given working value and consequently give the limiting value of r in equation (18).

Tables giving the resistance to torsion of various groups of rivets have been prepared by E. A. Rexford and are published by the Engineering News Publishing Co.

68. Flange Rivets. Ordinary Method of Computation of Pitch. Since it is through the rivets that stress is transmitted into the flanges it is evident that the number of rivets needed varies with the rate of increase of the flange stress, which depends upon the rate of increase of the moment. It follows that the rivet pitch (the pitch is to be considered the distance apart of the rivets measured along the flange, i.e., in the direction of the stress) may increase in direct proportion to the rate of increase of the moment. In a girder supported at the ends and carrying a uniform load the curve of moments is a vertical parabola, with its vertex on the vertical line passed through the centre of the span, hence the rate of increase of the moment is a maximum at the end and a minimum at the centre and the variation in rivet pitch should conform to this. This is also approximately true for such girders when loaded with other than uniform loads.

A knowledge of the variation of the pitch is, however, insufficient; it is necessary to determine the pitch itself. The following method of doing this is obvious. Compute the total stress in the flange at any section and compute also the total stress at a section one inch from the first; the difference between the two stresses gives the increase in flange stress per longitudinal inch at that portion of the girder and this increase must be carried into the flanges through the rivets. If one rivet can carry p lbs. and if the increase in flange stress is x lbs., the proper rivet pitch to use in that portion of the girder is $\frac{p}{x}$.

If the portion of the bending moment carried by the web be neglected in the determination of rivet pitch, this error being small and on the safe side, the increase in flange stress in the ordinary plate girder may be found by dividing the increase in moment by the distance between the centres of gravity of the

flanges. It has already been stated that the first derivative of the moment equals the shear and as it also equals the increase in moment, it follows that the rate of increase in flange stress at a given section equals the shear at that section divided by the distance between centres of gravity of the flanges.

The following formula for rivet pitch in the flanges of a girder may therefore be given:

$$p = \frac{R}{\sqrt{\left(\frac{V}{h}\right)^2 + m^2}} \qquad (19)$$

in which p = maximum allowable pitch in inches at section under consideration.

R = allowable stress on rivet in pounds (usually the value of rivet in bearing on web).

h = distance between centres of gravity of flanges in inches.

V = maximum external shear on given section in pounds.

m = load per lineal inch supported directly by flange.

If there is no vertical load imposed directly upon the flange as is usual in the case of the bottom flange and in that of the top flange when the girder is loaded by floor-beams, $m = 0$.

R will be the smaller of the following two values. *a*. The product of the allowable bearing value per sq. in. by the diameter of the rivet in inches by the thickness of the web plate. *b*. Twice the product of the allowable shearing stress per sq. in. by the cross-section area of the rivet in sq. in.

Owing to the fact that if the distance between rivets be too great the different pieces in a compression member may wrinkle, it is customary to specify a maximum pitch not greater than 6 in. or 16 times the thickness of the thinnest plate connected. This restriction is frequently the controlling factor in determining the rivet pitch, and is commonly applied to tension members as well as compression pieces.

Equation (19) is applicable only to rivets through the vertical leg. These are the rivets which carry the stress into the flange. The rivets through the horizontal leg serve to transmit a part of this flange stress into the cover plates and in consequence may have a larger pitch. It is customary, however, to use the same

pitch for the vertical as for the horizontal rivets,[1] hence the method given is, in general, all that is necessary.

69. Flange Rivets. Precise Method of Computation of Pitch. The method represented by Eq. (19) is the approximate method of figuring rivet pitch which is generally used in plate girder design. In order to thoroughly understand the question of rivet pitch and to be able to properly figure the pitch in other cases which may arise, such as columns carrying bending, it is necessary to develop a more exact method. Such a method may also be well employed in investigating existing girders the strength of which may be in doubt. To obtain such a method the formula for horizontal shear may be used.

Referring to Fig. 93, it is evident that the function of the rivets at a is to prevent the flange angles from sliding along the web; that is, the rivets must resist the longitudinal shearing tendency of the angles. Hence, if this tendency can be computed the rivet pitch necessary to withstand it can be determined. This computation can be easily made by multiplying the intensity of the longitudinal shear at the bottom of the angles by the thickness of the web, using for Q in the determination of the intensity, the statical moment about the neutral axis of the girder of the angles and cover plates combined, but not of the portion of the web included between the flange angles, since the stress in this is carried by the web itself. This gives the shearing force per longitudinal inch in the flange which equals the increase in flange stress, and this can be used as before in figuring the pitch. By the same method the correct pitch for the rivets at b can be determined by computing Q for the cover plates only.

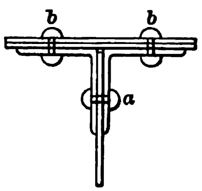


FIG. 93.

70. Flange Rivets. Example in Computation of Pitch. To illustrate the application of these methods the girder shown in Fig. 83 will be considered.

Problem. Determine the rivet pitch required at a section where the shear is 300,000 lbs., assuming that at this section all the cover plates are needed.

Solution. The rivets through the vertical legs of the angles will first be considered, assuming that the outer force is applied directly to the web and not through the flange. Let the bearing value per

[1] At the ends of the cover plates it is customary to place the vertical rivets at a small pitch for a distance of one or two feet to ensure that the stress may be properly carried into the plate.

square inch of the rivets be taken as 24,000 lbs. and the shearing value as 12,000 lbs. The strength of the rivet will then be limited either by bearing on the ½-in. web, which equals $\frac{1}{2}\times\frac{7}{8}\times 24{,}000=10{,}500$ lbs., or by double shear, which equals $0.60\times 12{,}000\times 2=14{,}400$ lbs. As the bearing value is smaller it should be used.

Increase in flange stress per linear unit:

Approximate method: $$\frac{300{,}000}{90.4}=3320$$

Exact method:

$$\frac{VQ}{I}=300{,}000\,\frac{16.88(45.25-1.78)+30.0(45.25+0.94)}{222{,}295}=2850$$

Since one rivet can carry 10,500 lbs. the required pitch by the approximate method is

$$\frac{10{,}500}{3320}=3.16'', \text{ or, say, } 3'',$$

and by the exact method $\frac{10{,}500}{2850}=3.67''$, or, say, $3\frac{1}{2}''$.

It is evident that the approximate method is decidedly on the safe side in this case.

Were the required pitch less than three diameters of the rivet it would be necessary to locate the rivets in two rows as shown in Fig. 94, where a pitch of 2 ins. is assumed.

To determine the pitch of the vertical rivets the exact method should be used. The increase in flange stress per inch is

$$\frac{30\times(45.25+0.94)}{222{,}295}\times 300{,}000=1870 \text{ lbs.}$$

The value of each rivet in this case is evidently its strength in *single* shear, but as there are two vertical rivets in each cross-section the pitch may be obtained by dividing the value of one rivet in *double* shear by the increase in flange stress per inch. This gives

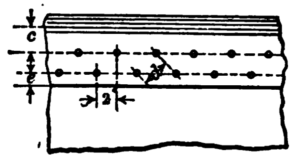


FIG. 94.

$$\frac{14{,}400}{1870}=7.7'', \text{ or, say, } 7\frac{1}{2}''.$$

As this exceeds 6″ (see Art. 68) the pitch of these rivets should be made 6″ or less.

It will be noticed that the pitch is the distance between rivets measured along the axis of the angle. The vertical distance between the rows of rivets must be sufficient to make the

distance d equal to or greater than three times the diameter of the rivet or $2\frac{5}{8}$ ins. for a $\frac{7}{8}$ in. rivet. The distance e should not be less than $1\frac{1}{4}$ ins., as already noted, and should preferably be $1\frac{1}{2}$ ins. or $1\frac{3}{4}$ ins. The distance c is determined by the amount of room needed for driving the rivet. The standard values for different angles are given in the steel-makers' handbooks.

This example shows that the pitch of the vertical rivets may be considerably greater than that of the horizontal rivets. It is, however, usually made the same for practical considerations.

There remains one other case to be treated—that of a girder supporting a load on the upper flange. To illustrate the method required for this case let the girder shown in Fig. 81 be considered, and let it be assumed that this girder is a railroad bridge stringer with ties resting directly upon the top flange. Let it also be assumed that the maximum wheel load crossing the stringer is 24,000 lbs.; that the maximum end shear including impact is 100,000 lbs.; and that the pitch of the rivets at the end of the stringer is to be determined. With the allowable unit stresses previously used the limiting value of the rivet is $\frac{7}{8}\times\frac{1}{2}\times 24{,}000 = 10{,}500$ lbs. Using the approximate method the increase in flange stress is found to be $\frac{100{,}000}{26.27} = 3800$ lbs. per linear inch. This value must be combined with the vertical load, carried by the rivets. Since the rail has considerable strength as a beam, it is evident that a wheel load will not be carried entirely by one tie, but will be distributed over several. The common assumption is that one wheel load is distributed over three ties. If this assumption be made one wheel load will be distributed over the rivets in the space m, Fig. 95. If the ties are 8 ins. wide and spaced 6 ins. apart in the clear, three ties will occupy a total distance of 42 ins., hence the vertical load per inch which the rivets must bear is $\frac{24{,}000}{42}$ or 570 lbs., neglecting the dead weight, which is so small compared with the live load as to be

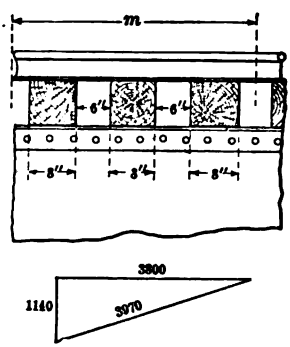


FIG. 95.

negligible. To allow for impact this value should be doubled since these rivets are more directly affected by the shock of the locomotive than any other portion of the structure. To obtain the rivet pitch, it is therefore necessary to divide the value of one rivet by the resultant of 3800 and 1140. This resultant may be obtained quickly and with sufficient accuracy by the graphical method indicated in Fig. 95. Its value is found to be 3970, hence the proper pitch at the end is $\frac{10,500}{3970} = 2.65''$ or, say, $2\frac{5}{8}''$.

This pitch can be used without difficulty for the girder under consideration since two rows of rivets should always be used in a 6 in. angle leg, and the actual distance apart of the rivets will, therefore, be considerably greater than the nominal pitch. Were the 4 in. leg vertical instead of the 6 in. leg, a pitch of $2\frac{5}{8}$ ins. could be used but this is the minimum allowable value, and the adoption of the minimum value except where unavoidable is not recommended, a better plan being either to increase the depth or thickness of the web to permit larger pitch, or else to use a wider legged angle with two rows of rivets. It frequently happens that the determination of the flange section of girders is materially influenced by the question of rivet pitch, and the experienced designer will always look into this before selecting flange angles.

71. Direct Web Stresses. It has previously been shown that the intensity of the horizontal shear at any point in the web of a

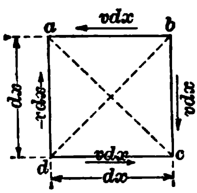


FIG. 96.

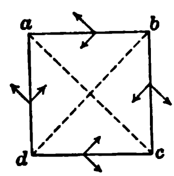


FIG. 97.

girder equals the intensity of the vertical shear and that these reach their maximum values at the neutral axis. Consider again an infinitesimal prism at the neutral axis. The shearing forces acting on this prism are shown in Fig. 96. These forces will develop internal forces of tension and compression, the value of which may be found as follows:

Let the thickness of the prism at right angles to the paper be unity, and let v equal the intensity of the shear. Then the total shearing force on each side $= vdx$. Resolving these forces into components the value of each is found to be $\frac{vdx}{\sqrt{2}}$, acting as indicated in Fig. 97. The effect of these components is to produce on the diagonal plane bd a total tension $= \frac{2vdx}{\sqrt{2}}$. Since the length of $db = \sqrt{2}dx$ the intensity of the tension on it is $\frac{2vdx}{\sqrt{2}\sqrt{2}dx} = v$.

In the same manner a compressive force may be shown to act on ac the intensity of which is also v. It therefore follows that at the neutral axis there exist both a tension and a compression acting upon planes at right angles to each other and at 45° to that axis, and that the intensity of these forces is equal to that of the shear. If the prism in question had been taken above or below the neutral axis these conditions would have been modified somewhat through the introduction of direct fibre stresses. The effect of the shear in producing direct stresses would not be changed, that is, the shearing forces would develop direct stresses as before, but the final value of the tension or compressign upon any plane would have to be obtained by combining the direct stresses due to shear and the direct fibre stresses due to bending.

The expression for the maximum direct stress for this more general case is developed in books on mechanics, and is as follows:

$$p' = \frac{p_x + p_y}{2} \pm \tfrac{1}{2}\sqrt{4v^2 + (p_x - p_y)^2}.$$

In this equation p' = intensity of the maximum direct stress occurring at a given point on any plane, p_x and p_y are the intensities of the direct stresses acting at the same point on two rectangular planes passing through the point, and v is the intensity of the shear on each of these two latter planes.

Fig. 98 illustrates this condition. It is evident that if point a is at the neutral axis of a beam subjected to bending but not to direct stress, p_x and p_y are both zero and $p' = v$.

The expression for the angle θ between the x plane and the plane upon which the maximum intensity occurs is also derived in mechanics, and is as follows for a beam subjected to bending only: $\tan\theta = \frac{p'}{v}$.

At the neutral axis of such a beam or girder $v = p'$, hence $\theta = 45°$.

That tension and compression act as shown in Fig. 97 is also evident from the distortion produced by the shearing forces. It is plain that under the action of horizontal and vertical shear the prism which is rectangular when unstressed will take the shape shown, greatly distorted, in Fig. 99, and hence the line *ac* will be lengthened, and the line *bd* shortened. These changes can be produced only by tension and compression at 45° to the axis.

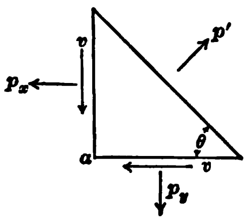


FIG. 98.

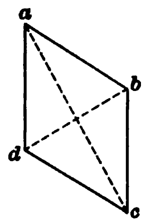


FIG. 99.

In plate girders the existence of this compression at 45° to the axis is of considerable importance since, if it be not recognized and proper means taken to provide for it, failure will occur through sidewise buckling of the web. To prevent such failure the web must be made of such thickness that there will be no danger of excessive compression, or else the buckling tendency must be restrained by other means. The latter is the common method, and is accomplished by the use of stiffeners in the form of vertical angles riveted to the web and extending from top to bottom of the girder. Sometimes, however, it is more economical of material as well as of labor, to increase the web thickness rather than to use stiffeners. In reinforced concrete beams the diagonal tension is an important factor since concrete is very weak in tension and means must generally be taken to provide against failure by rupture at 45° to the axis, either by steel rods placed at approximately 45° to the axis, or by vertical stirrups.

The combination of the direct stress due to shear with that due to bending gives a resultant compression in the web acting in the direction indicated by the dotted line in Fig. 100. The shape of this line is dependent upon the relative value of the shear and the direct stress. Its ordinates are plotted from axis mn. At the end of a girder where the shear is a maximum and the bending moment a minimum, it would lie at about 45° to the axis throughout its entire length. At the section where the moment is a maximum the shear is zero, hence at this section there is no direct stress in the web at the neutral axis, and the direct stress above or below this axis is parallel to it.

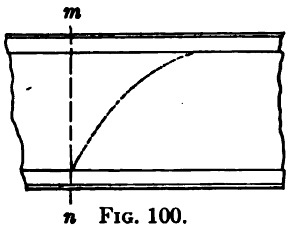


FIG. 100.

72. Web Stiffeners. The subject of stiffener spacing is complicated and no accurate theory has yet been developed. Experimental results have been inconclusive, but indicate that the ordinary methods of practice are safe if not precise. The only theoretical method that seems rational is to treat a strip of the web as a column and to make an assumption as to the influence of the remainder of the web upon this column.

By this method an equation can be deduced for the distance apart of the stiffeners which should be in rational form, and which, if it does not give results exceeding the limits of good practice, may be used with security.

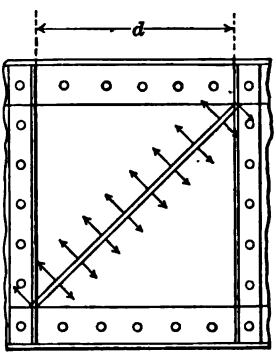


FIG. 101.

Let Fig. 101 represent a portion of the web near the end of a girder where the shear is a maximum. Since the bending moment at the ends of the girder is small the direct web stresses act at approximately 45° throughout the entire depth of the girder, hence the strip of web to be considered is taken at a 45° slope. Its length is restricted by the flange angles and it is partially restrained against sidewise buckling by direct web tension at right angles to its axis as indicated in the figure by arrows:

Let unity = width of strip.
t = thickness of web.
l = length of strip.
d = distance apart of stiffeners in clear.
r = radius of gyration.
I = moment of inertia of strip.
A = area of strip.

Then $\quad l=d\sqrt{2} \quad$ and $\quad r=\sqrt{\frac{I}{A}}=\sqrt{\frac{1}{12}\frac{t^3}{t}}=\frac{t}{\sqrt{12}}.$

A column formula of the form $\frac{P}{A}=16{,}000-c\frac{l}{r}$ will now be applied to the strip. Since c for the ordinary unsupported pin-ended column may be safely taken as 70, it would not seem unreasonable to reduce this materially here since the column is fixed at the ends by the flange angles and held sidewise throughout its entire length by the direct web tension. The amount which it should be reduced to allow for these restraining influences is unknown, but the value $c=25$ will be adopted as a conservative value, giving results which do not exceed the limits of ordinary practice. Substituting this value for the constant and expressing the value of $\frac{l}{r}$ in terms of t and d gives the following equation:

$$\frac{P}{A}=16{,}000-25\frac{d}{t}\sqrt{24}=16{,}000-120\frac{d}{t} \text{ (very nearly)},$$

in which $\frac{P}{A}$ equals the allowable intensity of axial compression in the strip. For the case in question this equals the shearing intensity, v, per square inch, hence the formula may be written thus,

$$v=16{,}000-120\frac{d}{t}. \qquad (20)$$

(Formula (20) should be used only when the live shear has been properly increased for impact.)

This formula may be used in either of the following ways:

1. To determine distance apart of stiffeners for a given shear and web thickness.

2. To determine the required web thickness for a given shear and distance apart of stiffeners or for the case where no stiffeners are used.

It should be noted that stiffeners placed further apart than the clear distance between the flange angles would not reduce the length of the strip shown in Fig. 101, and hence would theoretically be of no service at the ends of the girder where the shear has its maximum value. It is, however, customary to use stiffeners on all except shallow girders in order to stiffen the girder during fabrication and transportation, and a common requirement is that the maximum clear distance between stiffeners shall be the depth of web plate between flange angles, and shall not be greater than 5 ft.

The following example illustrates the method of using this formula.

Problem. Determine the required spacing of web stiffeners in the following girder:

Depth, 40½″ back to back of angles.
Web, 40″×½″.
Flange angles 6″×4″×½″ with 4″ leg vertical.
Maximum shear (live, dead, and impact) = 120,000 lbs.
Solution.

$$v=\frac{120,000}{\frac{3}{4}\cdot 40\cdot \frac{1}{2}}=16,000-120\frac{d}{\frac{1}{2}}. \qquad \therefore \quad d=33 \text{ ins.}$$

As the clear distance between the flange angles is 40½″−8″=32½″ it is evident that no stiffeners are needed, although the girder is just on the line. Had the web been thinner than ½″, stiffeners would have been required. For example, with a ⅜″ web, stiffeners would be required at intervals of 16⅔″ in the clear.

For that portion of a girder where the bending moment is large and the shear relatively small the conditions in the web differ materially from those assumed in developing formula (20). For example, at the point of maximum bending moment the shear is zero, and in consequence no web compression exists in the half of the girder between the neutral axis and the tension flange, while the web compression in the other portion of the girder is parallel to the flange and increases in intensity as the distance from the neutral axis increases. Between the section of maximum shear and that of maximum moment the condition varies from that assumed in developing the formula to that just

stated. While it is evident that the formula does not apply very closely to all these conditions, the fact that it gives a gradually increasing distance apart of the stiffeners as the shear diminishes, is probably consistent with actual conditions.

The size of intermediate stiffeners cannot be determined theoretically. A good rule is to make the outstanding leg equal to or greater than two inches plus one-thirtieth the depth of girder. The other leg should be of sufficient width to permit of proper riveting.

There is perhaps no point in plate girder design upon which engineers differ so greatly as that of stiffener spacing, and carefully conducted experiments are greatly needed to establish the necessary constants. The writer claims no special merit

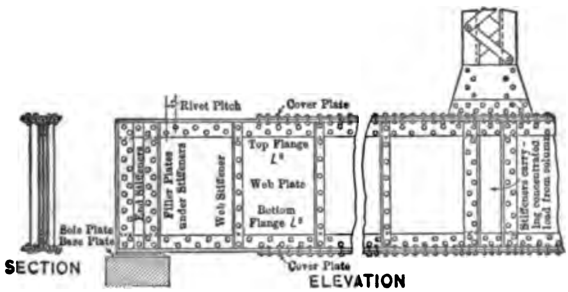


FIG. 102.

for his formula other than that it is derived from the column formula in common use at the present day, and that it gives conservative values.

The ratio between the unsupported length and radius of gyration of the column strip shown in Fig. 101 should be restricted, as otherwise the column formula used would be inapplicable.

If this ratio be restricted to 300, the corresponding value of $\frac{d}{t}$ is 60, a commonly specified limiting value for girders without stiffeners. While the ratio between the allowable unsupported length and radius of gyration may seem unduly high it should be remembered that this column is quite different from the ordinary column in being held throughout its entire length by the web tension.

In addition to the angles required to stiffen the web against buckling, stiffener angles should be used at all points where

concentrated loads of considerable magnitude are applied to the girder, in order to transmit these loads into the web without over-stressing the flange rivets. The design of such stiffeners consists in selecting angles of sufficient area in the outstanding legs to withstand the load without crushing and with sufficient total area to carry the applied load as a column, using the formula of Art. 18, and considering the unsupported length to be approximately one-half the depth of the girder. The number of rivets necessary to transmit the load into the web must also be determined, the value of the rivet being limited either by bearing on the web or by double shear. Both types of stiffeners are indicated in Fig. 102.[1]

73. Flange Plates. Flange plates are used to increase the flange area and thereby give a variable and more economical flange. It is not considered good design to use many cover plates. In general the total area of cover plates should not exceed one-half the total flange area, unless the largest sized angles are

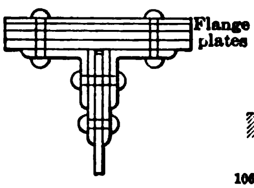


FIG. 103.

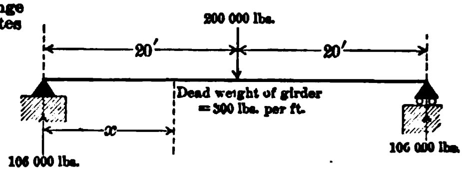


FIG. 104.

used. As the length of rivets should be limited in order to ensure good results, the thickness of the metal in the flanges should not exceed $4\frac{1}{2}$ ins. In case a larger flange area is required, vertical flange plates may be used, as shown in Fig. 103, or a box girder.

To determine the proper location of the ends of a cover plate it is necessary to equate the bending moment which the girder can carry without the cover plate to the external bending moment at the end of the plate.

The following example serves to illustrate this method:

Problem. How far from the ends of the girder shown in Fig. 104 may the ends of the cover plates be located? Girder to consist of a $48'' \times \frac{5}{16}''$ web, $6'' \times 6'' \times \frac{5}{8}''$ flange angles, and two $16'' \times \frac{1}{2}''$ cover plates on each flange. Allowable fibre stress = 16,000 lbs. per sq. in.

[1] For the results of experiments on the buckling of plate girder webs see article by Turneaure in the "Journal of the Western Society of Engineers for 1907," Vol. XII.

Solution. Effective area of the tension flange members:

		sq. in.
Two angles, $6'' \times 6'' \times \frac{5}{8}''$ at	$7.11 = 14.22 - 2.50 =$	11.72
Two plates, $16'' \times \frac{1}{2}''$ at	$8.00 = 16.00 - 2.00 =$	14.00
Web $\frac{1}{12} \cdot 48 \cdot \frac{9}{16}$	$=$	2.25

To locate end of outside cover plate proceed thus:
Effective flange area after plate is cut $= 11.72 + 7.0 + 2.25 = 20.97$ sq.ins.
Distance from back of angles to c.g. of flange

$$= 1.73'' - 8'' \times \frac{1.73 + 0.25}{22.22} = 1.0''$$

$$h = 48.5'' - 2.0 \text{ ins.} = 46.5''$$

Bending moment which girder can carry with one cover plate on flange.

$$= \frac{20.97 \times 16{,}000 \times 46.5}{12} \text{ ft.-lbs.}$$

$$= 1{,}300{,}000 \text{ ft.-lbs.}$$

Let x = distance in feet from end of girder to point where plate should begin.

$$\text{Bending moment at } x = 106{,}000x - \frac{300x^2}{2}.$$

$$\therefore \quad 106{,}000x - \frac{300x^2}{2} = \frac{20.97 \times 16{,}000 \times 46.5}{12} = 1{,}300{,}000.$$

The value of x as determined from this equation is 12.5 ft.

The actual length of the cover plate should be somewhat longer than the theoretical length, in order that its stress may be properly carried into it. A foot is usually allowed at each end for this purpose. If this allowance be made, the cover plate in question would begin 11.5 ft. from the end of the girder and its length would be 17 ft.

The value of x for the cover plate nearest the flange is given by the following expression:

$$106{,}000x - \frac{300x^2}{2} = \frac{13.97 \times 16{,}000 \times 45}{12}.$$

In the case of girders subjected to moving concentrated load systems the following graphical method may be used to advantage.

Plot the span and the external bending moments at each panel point or at correspondingly frequent intervals in the case of a deck girder without floor beams, connecting these latter points by a smooth curve, which will be the curve of bending moments. This curve is practically a series of straight lines and may be so used with safety if desired, the influence of the weight of the girder being offset by the fact that a straight-line curve for mov-

ing concentrated loads gives excess moments throughout except at panel points. (See Art. 49.) Compute the allowable moment, M,[1] by the application of formula (17) for the controlling conditions, viz., no cover plate; one cover plate on each flange; two cover plates on each flange; and so on up to the maximum number of cover plates used less one. Plot these moments to the same scale as the external bending moments. Since these moments for each case are constant throughout the length of the girder, each may be represented graphically by a straight line parallel to the girder axis, the points of intersection of which with the curve of bending moments locate the ends of the cover plate.

This method is shown for a girder with two cover plates by Fig. 105. M_a, M_b and M_c are the external bending moments;

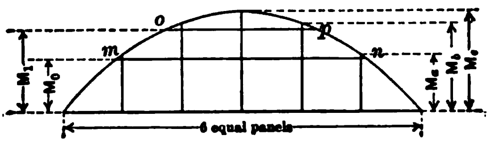


FIG. 105.

Mo is the allowable moment without cover plates; M_1 with one cover plate. (The moment with two cover plates need not be plotted.) The cover plate nearest the flange should extend from m to n, the outer cover plate from o to p. These are theoretical lengths and the actual plate should be made somewhat longer, as previously stated.

It should be noted that the curve of moments for end-supported girders carrying uniform loads is a parabola and the ends of the plates may be located by means of the following parabolic formula, which may also be used without serious error for girders carrying moving loads

$$c = L\frac{a}{A},$$

where c = length of cover plate in feet,

L = span in feet,

A = gross area of compression flange or net area of tension flange in square inches, at center of girder.

a = gross area of cover plate to be cut plus that of all plates further from the flange angles. Net area is used for tension flange.

[1] The allowable moment which the girder can carry may be called the moment of resistance.

The flange width is an important feature and should be carefully considered in selecting angles and plates. It is common in railroad bridge practice to specify that the compression flange should be supported laterally at intervals not greater than twelve times its width, this being accomplished in half-through bridges by brackets attached to the floor beams and in deck spans by cross frames and horizontal bracing. In case it is necessary to deviate materially from this rule the flange should be figured as a column. For the sake of appearance it is usual to select cover plates of sufficient width to project slightly beyond the flange angles on either side. They should, however, project not more than 2 ins. For example, flanges with 6″×6″ angles should have plates not less than 13 ins. and not more than 16 ins. in width. Plates with a width in even inches should preferably be chosen.

74. Connection Angles and Fillers. It is necessary either to use fillers under plate girder stiffeners, or else to crimp the stiffener angles over the flange angles. There is but little difference in the cost of the two methods, but the former is generally preferred.

One objection to the use of fillers is that unless the filler is riveted to the web plate by an independent row of rivets, thus becoming practically a portion of the web (this type of filler is frequently called a tight filler), the rivets connecting the stiffener to the web are reduced in strength since they have to carry stress through the loose filler plate and thus are subjected to some bending. This is of no importance in intermediate stiffeners which serve merely to stiffen the web, but should be considered in the case of stiffeners carrying a concentrated load into the web. In such cases if loose fillers are employed an excess of rivets, say 50 per cent, should be used.

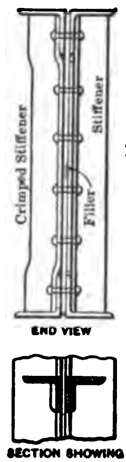


FIG. 106.

The use of tight fillers is also advisable in some cases in order to increase the bearing value of rivets which otherwise would be limited by bearing on the web instead of by shear. The following example illustrates this:

Problem. Determine whether sufficient rivets are used in the connection of stringers to floor beams shown in Fig. 107.

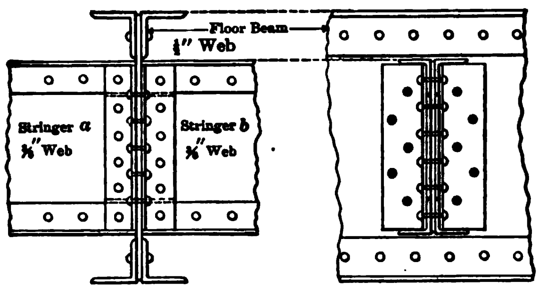
Allowable unit stress per sq. in. upon rivets:

	Bearing	Shear
Machine	24,000	12,000
Hand	18,000	9,000

For $\frac{7}{8}$-in. rivets above units give the following working values:

Machine-Bearing on $\frac{3}{8}$-in. plate = 7875 lbs. Shear = 7200 lbs.
Hand- " " $\frac{1}{2}$-in. " = 7875 " " = 5400 "

Assume that the rivets shown in Fig. 107 are all that can be used in the angles. Field rivets, which are hand rivets, are shown thus (.).



Maximum end shear: stringer a = 40,000 lbs.; stringer b = 30,000 lbs.
Maximum reaction on floor beam from both stringers = 65,000 lbs.

FIG. 107.

Solution. To carry to the hitch angles the shear of 40,000 lbs. in stringer a, there are required $\frac{40,000}{7875} = 5.1$ or 6 rivets to connect the stiffener angles to the web.

As indicated in the figure, the largest number of rivets that can be used is 6, but it is inadvisable to count upon those in the flanges, which are frequently fully stressed by the flange stress, and additional rivets should be added if the filler is to be a loose one, hence it is necessary either to use wider hitch angles with two rows of rivets, or else a wide filler to increase the bearing value of the connection rivets. The latter would be cheaper and consequently advisable. If the filler, therefore, be made wider and connected to the web by two extra rivets an additional stress equal to that which two rivets can carry can be taken from the web into the filler and by that carried into the rivets connecting the stiffeners. As these rivets would, however, have to carry a considerable bending moment in addition to the direct shear it is advisable to make a liberal allowance, hence it would be well in this case to use in the fillers four rivets placed directly opposite those in the stiffeners. The stringer would then be as shown in Fig. 108.

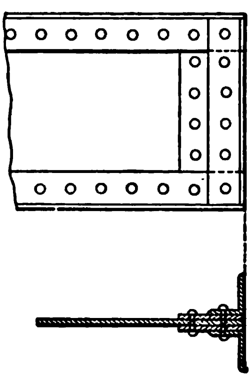


FIG. 108.

One other point yet remains to be considered, viz., the shearing value of the rivets. The connection has so far been designed to carry the stress from the web plate into the rivets. Can the rivets carry this stress into the angles? As the thickness of the angles is not restricted sufficient bearing area can be obtained, but can the rivets carry the stress without shearing off?

As the rivets are in double shear there are needed $\frac{40,000}{14,400} = 2.8$ or 3 rivets, hence the number needed for bearing is also sufficient for shear, otherwise it would be necessary, despite the wide filler, to use wider stiffener angles.

The connection of stringers to floor beam may be treated in a similarr manner. Ten hand rivets are shown. These have to carry in single shear the maximum shear in a single stringer, i.e., 40,000 lbs. They also have to carry 65,000 lbs. in bearing upon the web.

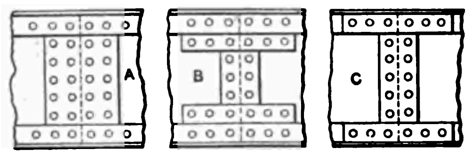
$\frac{40,000}{5400} < 10$, hence there are enough in shear;

$\frac{65,000}{7875} = 8(+)$ or 9, hence there are also enough in bearing.

75. Web Splices. Owing to the limited length of plates obtainable it is frequently necessary to splice the webs of long and deep girders. Fig. 109 shows several methods of making such splices.

Of these type A is best in appearance and is recommended for use.

The design of such splices requires two distinct operations, viz., the determination of the size of the splice plates, and the



Web Splices.

FIG. 109.

determination of the number and location of the splice rivets. The former question involves the selection of plates that are of sufficient strength to carry not only the shear at the section where the splice is to be located, but also the bending resistance of the web as given by formula 17, viz., $\frac{1}{12}$ its gross area multiplied by the product of the allowable unit stress and the distance between centres of gravity of flanges. For a splice of the type shown by A, Fig. 109, both of these considerations are usually

satisfied by plates of the minimum allowable thickness, although for thick web plates or shallow girders the thickness of the plates should be carefully computed by the method used in the following example. The width of the plates is usually determined by the number of rivets needed, and requires no computation. The rivets must be sufficient to carry the shear and bending moment which the splice plates are required to resist. Their computation involves the application of the method given in Art. 67 for the strength of rivets in torsion.

One of the well recognized and important rules of good design is to so proportion the member that it will be equally strong at joints and other critical sections as in its main portion. The application of this rule to the design of web splices involves making the splice of sufficient strength to carry all the shear and bending moment which the web plate is capable of carrying. For many girders this would give excessive strength since the web is not called upon to resist maximum moment and shear simultaneously. It is, however, a safe rule to follow in all cases and should not be deviated from unless the location of the splice can be so fixed that it will surely come at a point where it will not be subjected to maximum conditions of both kinds simultaneously.[1]

The example which follows illustrates the design of a web splice for a girder, the web plate of which is supposed to be fully stressed in shear and bending.

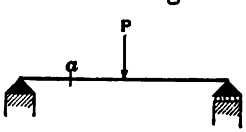


FIG. 110.

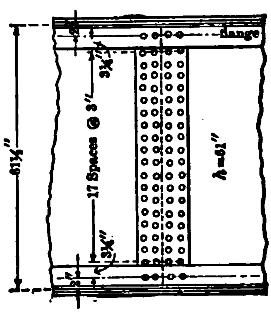


FIG. 111.

Problem. Design a web splice for a girder with a $6'' \times \frac{7}{16}''$ web, $6'' \times 3\frac{1}{2}'' \times \frac{9}{16}''$ flange angles with $3\frac{1}{2}''$ legs vertical, and two $16'' \times \frac{3}{8}''$ cover plates on each flange, using the unit values specified in Art. 18 and $\frac{7}{8}''$ rivets.

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[1] Such a condition could exist at section *a*, in the girder shown by Fig. 110, provided a cover plate stopped a little beyond this point.

Solution. First assume the number and location of rivets which can be used in one vertical row. As shown by Fig. 111, eighteen rivets may be used, spaced 3 ins. apart. Had the minimum allowable spacing of three times the rivet diameter been adopted, a few more rivets could have been inserted in a row, but it is inadvisable to use the minimum pitch if it can be avoided.

Next assume that the minimum allowable thickness of material is $\frac{5}{16}$ in. and determine if this thickness will prove sufficient for the splice plates. If these plates are assumed to be fitted to the edges of the vertical legs of the flange angles their length will be $54\frac{1}{2}$ ins. and the net area of the two splice plates through a row of rivets will equal $2(54\frac{1}{2}-18)\frac{5}{16}=22.8$ sq.ins. The net area of the girder web equals $(61-20)\frac{7}{16}=17.9$ sq.ins., hence $\frac{5}{16}$ in. plates are ample to carry the shear.

To determine whether their strength in bending is sufficient, the allowable resistance of the girder web as used in formula 17 should first be determined. This equals $(\frac{1}{12}\cdot 61\cdot \frac{7}{16})sh=2.22\times 61.1\times s=135.62s$. If there were no rivet holes in the splice plates their resistance to bending would be given by the formula,

$$M=\tfrac{1}{6}sbh^2=\tfrac{1}{6}s\cdot(\tfrac{5}{8})(54\tfrac{1}{2})^2=309s.$$

As already stated the allowance for rivet holes is approximately equal to that obtained by using a coefficient of $\frac{1}{8}$ in the formula for M instead of $\frac{1}{6}$. Making this allowance gives the following value for the resistance[1] to bending:

$$M=\tfrac{1}{8}sbh^2=\tfrac{3}{4}\,309s=232s.$$

This value is much larger than necessary, hence the $\frac{5}{16}$ in. plates are of sufficient strength to carry bending and shear.

To determine the number of rows of rivets the allowable shearing and bending resistance of the girder web must be computed. These values are as follows:

Shear, $61\times\frac{7}{16}\times\frac{3}{4}\times 12{,}000=240{,}000$ lbs.

Bending, $135.62\times 16{,}000=2{,}170{,}000$ in.-lbs.

If two rows of rivets are assumed on each side of the splice the vertical load per rivet will equal $\dfrac{240{,}000}{2\times 18}=6666$ lbs.

The value of a rivet in bearing on the $\frac{7}{16}''$ web $=9187$.

The method of Art. 67 may now be applied, but is somewhat laborious and no essential error will be made if the resistance of each rivet to torsion be assumed to vary with its distance from the central axis of the girder instead of its distance from the centre of gravity of the group of

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[1] The actual effect of the rivet holes in the tension half of the splice plates in this case is to reduce the value of I for the gross area by the following amount: $\frac{5}{8}(25.5^2+22.5^2+19.5^2+16.5^2+13.5^2+10.5^2+7.5^2+4.5^2+1.5^2)=\frac{5}{8}(2180)=1362.$

The value of I for the gross area $=\frac{1}{12}\cdot\frac{5}{8}\cdot(54.5)^3=8431$;

hence the reduction made by using the coefficient $\frac{1}{8}$ instead of $\frac{1}{6}$ is ample.

rivets, and to act in a direction parallel to this axis. Making this assumption the resistance of the outermost rivet to bending will equal:

$$\sqrt{\overline{9187}^2 - \overline{6666}^2} = 6320 \text{ lbs.}$$

The value of I in formula 18 has already been computed for a half row of rivets. (See foot-note.) The resistance to bending of the two rows of rivets may now be written

$$R = 2\left(\frac{6320}{25.5}\right)(2\times2180) = 2{,}161{,}000 \text{ inch-lbs.}$$

This is practically equal to the value previously found for the allowable web resistance, viz., 2,170,000 in.-lbs. Were the resistance of the rivets to bending materially less than the allowable bending moment of the web another row of rivets might be used or the splice located at a point where the web is not fully stressed in bending and shear simultaneously. A good location in such a case would be at a point a slight distance toward the centre of the girder from the end of a cover plate. At such a point practically all the cover plate area would be in excess and could be counted upon to make up the deficiency in the strength of splice.

The method of calculation illustrated by the previous example is not strictly accurate, and probably less so than for the cases given in Art. 67, where the number of rivets in a vertical row was much less. It should be noted that for such cases the distribution of the shear over the rivets is probably by no means uniform, the rivets near the neutral axis, where the shear is a maximum, probably carrying more than those nearer the outer fibres.

In practice it may be found desirable to use splice plates thicker than those required by computation. If the splice plate be used as a filler it should be as thick as the flange angles. It is, however, possible in such a case to make up the total thickness required by the use of a $\frac{5}{16}$-in. splice plate and a filler, an arrangement frequently used.

76. Flange Angle Splice. In very long girders it is frequently necessary to splice the flange angles. When this is to be done only one angle in each flange should be spliced at a section. A common practice is to splice the top angle on one side of the girder and the bottom angle on the other side at a section a little to one side of the centre of the girder, and to reverse this process for a corresponding section on the other side of the centre. The splice should always be made by another angle the cross-section area of which should be equal to that of the angle to be spliced. In order to simplify the construction the splice angles for the

tension flange should be exactly like those for the compression flange, hence the net area of the splice angle should equal the net area of the main angle.

In order to obtain a splice of neat appearance and which answers the above requirements it is usually necessary to select an angle with the same width of legs as the main angle, but $\frac{1}{16}$ in. or $\frac{1}{8}$ in. thicker, and to plane off the projecting legs so that they may be flush with the main angle.

The following example illustrates this: Determine the splice angle required for a $6'' \times 6'' \times \frac{1}{2}''$ flange angle. The net area of the main angle $= 5.75 - 1.00 = 4.75$ sq.ins. The net area of a $6'' \times 6'' \times \frac{9}{16}''$ angle planed to fit the $6'' \times 6'' \times \frac{1}{2}''$ angle $= 6.44 - 1.12 - 2(\frac{1}{16} \times \frac{1}{2}) = 4.76$ sq.in., hence this angle has just the right area and should be used.

Fig. 112 shows by cross-hatching the portion of the angle to be cut off. The outer corner must also be rounded off as indicated to fit the fillet of the main angle.

The number of rivets required in the splice angle may be determined, if there are no cover plates, by computing the stress which the angle can bear and dividing it by the value of one rivet, the strength of the rivet being generally limited by single shear. If the angles are equal-legged one-half the number of rivets needed should be used in each leg. If the legs are unequal

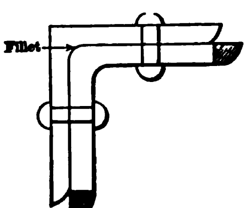


FIG. 112.

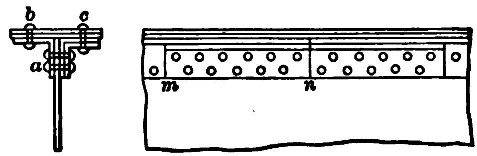


FIG. 113.

each leg should have its proportional part of the total rivets required; e.g., if a $6'' \times 4''$ angle is to be spliced and if 20 rivets are needed in all, put $\frac{6}{10} \cdot 20 = 12$ rivets in the 6-in. leg, and $\frac{4}{10} \cdot 20 = 8$ rivets in the 4-in. leg.

Cover plates are generally required in girders which require flange angle splices, and in such girders the number of splice rivets needed may be somewhat in excess of the number required to carry the total stress which the angles are good for, since the increment

in stress in a distance equal to the length of the splice angle must be taken care of. Referring to Fig. 113, it is evident that the horizontal rivets at *a* must carry from the flange into the splice angle in the distance *mn* one-half the increment in flange stress in that distance (the other half going through the same rivet to the flange angle on the left-hand side) plus one-half the stress in the main angle at *m* (since the angle is equal-legged). The rivets at *c* should be computed to carry the same amount, since it is proper to assume that all the increment in flange stress is carried by the cover plates, the angle being fully stressed before cover plates are added. The rivets at both *a* and *c* are limited usually by single shear, and should be designed accordingly. Since the splice would ordinarily be placed near the centre of the girder where the increment in flange stress is small, it is usually sufficient to determine the number of rivets required to splice the angle, assuming it to be stressed to its full value and to add one or two rivets to carry the flange stress increment. If no cover plates are needed it is unnecessary to consider the increment in stress since if the splice rivets be determined for the full value of the angle they will surely be sufficient to carry the stress in the angle at *m* plus the increment in *mn*.

77. Cover-plate Splice. A cover-plate splice may always be made by the addition of a splice plate of the same size as the plate to be cut, and the use of sufficient rivets to transmit the full stress from one plate to another, with the addition of a liberal percentage of extra rivets, say 33 per cent for each plate intervening between the plate to be spliced and the splice plate. Such a splice is shown in Fig. 114.

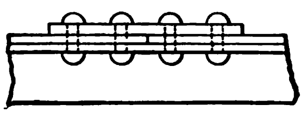


FIG. 114.

The disadvantage of long rivets, subjected perhaps to bending moment because of the intermediate plates, together with the unsightly appearance of such a splice, makes it desirable if the girder has more than one cover plate to splice one cover plate by means of another. This may be done by properly choosing the section where the splice is to be made.

This is illustrated by Fig. 115, in which the lower cover plate is to be spliced. If this plate be cut at *a*, where the top cover plate should begin, and if the top plate be extended to *b*, making the distance *ab* such that enough rivets can be put between *a* and *b* to carry the stress that the plate is good for, the splice will be

properly made. If the top plate be thinner than the bottom plate the splice would have to be located nearer the end of the girder

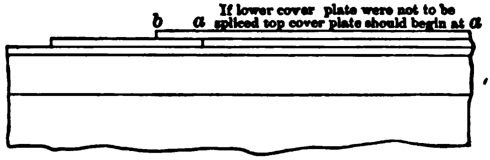


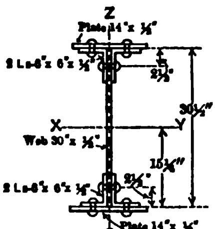
FIG. 115.

at a point where one cover plate of the thickness of the top plate would be just sufficient to carry the stress.

It will be observed that this method is based upon the transfer of all the stress from the end section of the plate to be spliced into the plate immediately above it. The intermediate section of the spliced plate instead of the upper plate will then take the additional increment of flange stress.

PROBLEMS

40. *a.* Compute $\frac{I}{c}$ for this girder with respect to the neutral axis and to the axis ZQ, and compute maximum fibre stress for a total uniformly distributed vertical load of 3200 lbs. per foot over the entire girder. Span = 40 ft.



PROB. 40.

b. Compute the maximum fibre stresses in both flanges for the loading given in *a*, using formula (17), Art. 62. Allow for holes for ⅞-in. rivet.

c. Compute by both the approximate and exact methods the required pitch of the horizontal and vertical flange rivets at end of top flange assuming all the load to be applied directly through the flange, and one cover plate on each flange to extend to end of girder. Use unit values given in Art. 18, and ⅞-in. rivets.

d. Determine distance from end at which the cover plates may be cut if desired.

e. Determine whether intermediate stiffeners are needed.

f. Determine the size of stiffeners needed on the girder to support the top flange under a concentrated load of 200,000 lbs. Allow 20,000 lbs. per square inch bearing on stiffeners and assume that their outstanding legs only are effective.

CHAPTER VI

SIMPLE TRUSSES

78. Trusses Defined. A truss is a structure consisting of separate bars designed to carry direct tension or direct compression. These bars are connected at their ends and occasionally at intermediate points, the points of connection being called joints. The connections are sometimes made by riveting the members directly together and sometimes by riveting them to a common steel plate, the truss in either case being called a *riveted truss.* The connections may also be made by fastening together with a large steel pin all the members meeting at a joint; such a truss is called a *pin truss.* The outer forces should be applied at the joints only, since the members are not intended to carry bending. This is accomplished by the use of floor beams in a bridge and purlins in a roof. As the depth of plate girders is limited by the available width of plates and by the inability to ship by rail single pieces wider than 10 ft. 6 in., or thereabouts, it is necessary to use trusses where economy or rigidity require greater depths. The common practice in the United States at the present time is to use beams or girders up to lengths of 90 or 100 ft.; riveted trusses above these lengths up to 150 or 175 ft.; and pin trusses above this length. The use of shorter pin-truss spans for railroad bridges has been given up because of lack of rigidity and consequent early wearing out of the bridge.

A typical pin truss is illustrated by Fig. 4.

79. Classification. All trusses may be divided into two general classes based upon the methods necessary for the determination of the stresses in the members; if these stresses can be determined by statics the truss is statically determined; otherwise it is statically undetermined. It should be noted that a truss may be statically undetermined with respect to the outer forces, i.e., the reactions cannot be determined by statics, and yet be statically determined with respect to the inner forces, and *vice versa.* The former is usually the case with draw bridges, the latter with the double intersection trusses sometimes used in simple span bridges.

80. Theory. The theory upon which the computation of truss stresses is based assumes that the centre of gravity lines of the members meeting at a point intersect at a point and are held together at that point by a frictionless pin. It follows that the stresses in the various members will all be direct stresses. That this deviates considerably from the truth for riveted trusses is

evident; the error in pin trusses is less, but not negligible, hence the common theory of trusses is by no means an exact one. The secondary stresses produced by resistance to motion at the joints are, however, small in well-designed trusses, as compared with the primary stresses, as the stresses computed by the above assumption may be called, and experience shows that for simple spans of ordinary length these primary stresses are sufficiently exact to be used in designs where the ordinary factor of safety is applied.

81. Methods. The methods necessary for the computation of the stresses in statically determined trusses are very simple, and consist merely of the application of the three equations of equilibrium to portions of the truss, these portions being chosen in such a way as to enable the stress in a given bar to be easily found. There are in common use three methods of accomplishing this result; the method of *joints*, the method of *moments*, and the method of *shears*. All of these are applications of the general method and differ only in detail. In the computation of a truss it is often advantageous to employ all three methods, choosing for each bar that which is best adapted to it. The method of joints is the most general of these methods and will be considered first.

82. Analytical Method of Joints Described. Fig. 116 represents a simple truss carrying a load at the apex. Let a section be taken around the joint at a and the remainder of the truss removed. As the entire truss is in equilibrium the portion enclosed by this section, shown by a circle in the figure, must also be in equilibrium, hence the two equations of equilibrium $\Sigma H=0$ and $\Sigma V=0$, will suffice to determine the unknown stresses in ab and ac the horizontal and vertical components of each of which are functions of the bar stress and are to each other as the horizontal and vertical projections of the bar. In a similar manner other portions of the truss may be cut off by similar sections taken at a sufficient number of joints to permit the determination of all the unknown stresses.

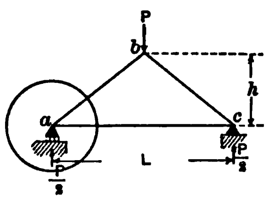


FIG. 116.

Fig. 117 shows the condition which exists at joint a, assuming that the stresses in the bars are axial stresses, this being in accordance with the general theory. It should be carefully observed that this method deals with the stresses in the bars rather than with the bars themselves.

Referring to Fig. 117 it is evident that as there are but two unknown forces, S_1 and S_2, the two equations of equilibrium, $\Sigma H=0$ and $\Sigma V=0$, will be just sufficient to determine these, and since all the forces meet at a point the equation $\Sigma M=0$ will be satisfied by any value of S_1 and S_2 and need not be considered. With the stresses in bars *ab* and *ac* thus determined, a section may next be taken at either joint *b* or joint *c*, and the stress in bar *bc* computed in a similar manner, thus completing the necessary computations for this truss.

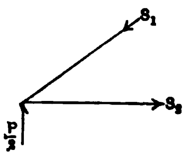


FIG. 117.

It should be observed that if the moment at a joint does not equal zero, the number of unknowns will be increased, and additional equations which can not be derived from statics would be necessary to obtain a solution.

83. Character of Stress. The determination of the character of the stress is often more important than that of its magnitude, as a bar designed for tension may fail if the stress is compression even if its magnitude is small. To determine the character of the stress in any bar it is sufficient to arbitrarily assume the direction of the stresses before applying the equations of equilibrium. If the solution gives a positive result for a stress it shows that this stress acts in the direction originally assumed.

In this connection it should be carefully observed that the *internal* stresses in a bar subjected to tension continually tend to pull the ends together; that is, to shorten the bar, hence tension in a bar always acts *away* from the joints at *both* ends, and *compression toward* the joints. Fig. 117 illustrates this. S_2 is shown acting away from the joint, that is, in tension; S_1 on the other hand is assumed in compression and is shown acting *toward* the joint. If the joint at the top of the truss should next be investigated it would be necessary to represent S_1 as acting toward that joint also, since the computations give a positive value for S_1, thus indicating that it is compression.

It is safer for the beginner to assume the stress in each bar as tension, or away from the joint. Positive values will then indicate tension and negative values compression. This is in accordance with the common but not universal convention of representing tension (which increases the length of a bar) by a *plus* sign.

84. Determinate and Indeterminate Trusses. For the truss under consideration there are three unknown bars. The stresses

in two of them have been determined by considering one joint only; the stress in the other may be found by taking either of the other joints. Since by taking both of the other joints there would be four equations and only one other unknown bar it would seem as if there were too many equations. This is a fallacy, however, as these equations must suffice to determine the unknown reactions as well as the unknown bars, since equilibrium of each joint involves equilibrium of the entire structure; that is, for this particular structure and in general for all planar structures which are statically determined with respect to the reactions there must be three more equations than there are bars. In other words, the six equations of joints for such a truss are not independent but are related in such a manner as to satisfy the three general equations of equilibrium for the truss as a whole, viz., $\Sigma X=0$, $\Sigma Y=0$ and $\Sigma M=0$, which may for most cases be replaced by the more common equations

$$\Sigma H=0, \quad \Sigma V=0, \quad \Sigma M=0.$$

There are therefore for the truss shown in Fig. 116 but three *independent* equations which can be used in determining the bar stresses, hence these stresses are determinate.

In general it may be said for all *planar* trusses which are statically determined with respect to the outer forces, that if n equals the number of joints, $2n-3$ equals the number of bars which the structure must have to be determinate. If it has more bars the stresses can not be computed by statics; if less it will not be rigid and will collapse except under special conditions. The same criterion must also be applicable to any portion of the truss. For non-planar trusses the conditions are somewhat different but will not be discussed here.

If it be desired to build a structure which because of the number of points of support or for other reasons would ordinarily be statically undetermined with respect to the outer forces, it may be possible to make the structure determinate in all respects by properly choosing the number of members. For example, in the case of a cantilever truss, a diagonal over a pier is sometimes omitted for this reason. In swing spans diagonals are often omitted or made very small in order to reduce the numbers of unknowns. If the unknown components of the reactions be four it is evident that there can be only $2n-4$ bars if the structure is to be made determinate.

85. Mode of Procedure. Analytical Method of Joints. In the solution of problems by the analytical method of joints the following mode of procedure should always be adopted:

1. Compute reactions.
2. Select a joint at which only two bars meet.
3. Assume the stresses in these bars to be tension, that is, to act away from the joint; and apply the equations of equilibrium. If the stress in either bar is found to be negative it indicates that the bar is in compression instead of tension.
4. Consider any other joint at which only *two* unknown bars meet and determine the stresses in these bars in the same manner and proceed thus until all the stresses have been determined.

86. Application of Analytical Method of Joints. The following numerical example has been worked out to show the application of this method:

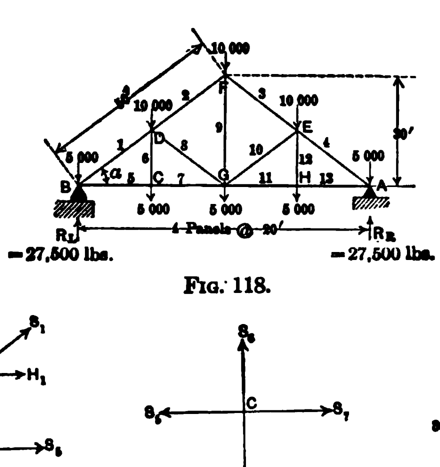


FIG. 118.

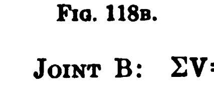


FIG. 118B.

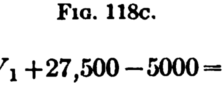


FIG. 118C.

FIG. 118D.

JOINT B: $\Sigma V=0$: $V_1+27{,}500-5000=0$,

$$V_1=-22{,}500 \text{ lbs.}$$

$$H_1=\frac{40}{30}V_1=-30{,}000 \text{ lbs.}$$

$$S_1=\frac{50}{30}V_1=-37{,}500 \text{ lbs.}$$

$$\Sigma\mathbf{H}=0:\ H_1+S_5=0,$$

$$S_5=+30{,}000 \text{ lbs.}$$

JOINT C: $\Sigma\mathbf{V}=0:\ S_6-5000=0,$

$$S_6=+5000 \text{ lbs.}$$

$$\Sigma\mathbf{H}=0:\ 30{,}000-S_7=0,$$

$$S_7=+30{,}000 \text{ lbs.}$$

JOINT D: $\Sigma\mathbf{V}=0:\ 22{,}500-15{,}000+V_2-V_8=0.$

$$\Sigma\mathbf{H}=0:\ 30{,}000+H_2+H_8=0.$$

But $$H_8=\frac{20}{15}V_8=\frac{4}{3}V_8,$$

and $$H_2=\frac{40}{30}V_2=\frac{4}{3}V_2.$$

$\therefore$ from above equations may be obtained the following:

$$7500+V_2-V_8=0,$$

$$30{,}000+\frac{4}{3}V_2+\frac{4}{3}V_8=0.$$

Solving, $V_2=-15{,}000$ lbs. $\quad H_2=-20{,}000$ lbs.

$V_8=-\ 7{,}500$ lbs. $\quad H_8=-10{,}000$ lbs.

$$\therefore\quad S_2=-15{,}000\times\frac{5}{3}=-25{,}000 \text{ lbs.}$$

and $$S_8=-7500\times\frac{25}{15}=-12{,}500 \text{ lbs.}$$

The computation for a bar such as 8 may sometimes be advantageously referred to other than horizontal and vertical axes. If, in this particular case, the X axis be taken along the upper chord BF, and the Y axis perpendicular to it, the condition at the joint will be as shown in Fig. 119. It is clear that in this case the value of Y_8 is given at once by the equation $\Sigma Y=0$, and equals $-12{,}000$ lbs.

$$\text{The actual stress in the bar} = -\frac{12{,}000}{\sin\theta} = -\frac{12{,}000}{2\sin a\cos a}$$

$$= -\frac{12{,}000\times 25}{24} = -12{,}500 \text{ lbs.}$$

It will be noticed that the stress in this case has been determined without reference to stresses S_1 and S_2, and is a direct function of the stress in bar 6 and the panel load at D.

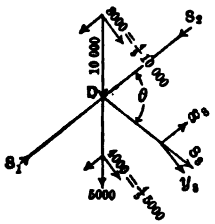


FIG. 119.

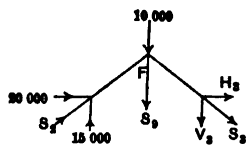


FIG. 119A.

JOINT F: $\Sigma \mathbf{V}=0$: $15{,}000-10{,}000-V_3-S_9=0.$

$\Sigma \mathbf{H}=0$: $20{,}000+H_3=0 \quad \therefore H_3=-20{,}000$ lbs.

But $V_3=\frac{3}{4}H_3,$

$\therefore \quad V_3=-15{,}000$ lbs.

hence $S_9=+20{,}000$ lbs.

and $S_3=-15{,}000\times\frac{5}{3}=-25{,}000$ lbs.

Since the truss is symmetrical and the loads are also symmetrical, the stresses in the bars of one-half the truss are identical with those in the bars of the other half, hence further computations are unnecessary.

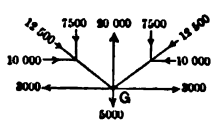


FIG. 120.

As a check consider joint G, at which forces will act as shown in Fig. 120. It is evident that these forces are in equilibrium, hence the stresses are checked to a certain extent. In practice further checks should be applied.

87. Graphical Method of Joints Described. The analytical method just given is perfectly general but too laborious to use in determining the stresses in all the bars of an ordinary truss, though it may be used with great advantage for certain specific members. A graphical method based upon the same principles is well adapted for many types of trusses, particularly roof trusses with non-parallel chords, and should be thoroughly understood. This method consists of drawing polygons of forces for each joint in succession, the polygons being so combined as to considerably reduce the labor which would be required if each joint were to be considered separately. The stresses in the bars can be obtained by scaling the sides of the polygons. Like other graphical processes, this method is less precise than analytical methods, and errors in scaling the stresses are easily made and difficult of detection. A closure of the figure, however, would indicate that no error of importance had been made in the graphical work.

88. Mode of Procedure. Graphical Method of Joints.

1. Draw a sketch of the structure to any suitable scale and show on it all the outer forces including reactions.

2. Designate all the forces and bars by letters so located that each force and each bar will lie between two letters and only two, as illustrated by Fig. 121.

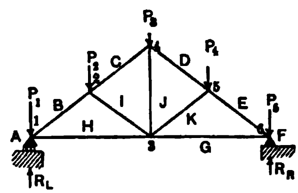


FIG. 121.

3. Draw a polygon of outer forces. This should be drawn to a scale of sufficient size to give the desired accuracy and the forces should be plotted in the order in which they are reached by going around the figure in a clockwise direction, and should be lettered at the ends by the letters in the order obtained by this clockwise rotation. This polygon should close if the reactions have been correctly determined.

4. Draw a triangle of forces for each joint, beginning at any joint where an outer force and two bars only meet, and proceeding thence, joint by joint, selecting the joints in such an order that at no joint will there be more than two undetermined forces to consider. The sides of these triangles representing the outer forces are the sides of the force polygon. The sides representing

bar stresses should be lettered at the ends by the letters obtained by going around the joints in a clockwise direction. The diagram thus drawn should form a closed figure.

5. Determine the magnitude and character of the bar stresses from the diagram. The magnitude of the stress in any member equals the length of the line of the diagram parallel to the bar in question measured to the scale of the force polygon; its character is determined by the order in which the letters are reached in going about any joint in a clockwise direction. For example, to determine the character of the stress in bar CI of Fig. 121, note that ci in the stress diagram, Fig. 122, acts downward to the left, as determined by the order in which the letters are reached in going around joint 2, hence the stress in CI also acts downward to the left, or toward the joint, since the bar is above the joint, and is therefore compression. A similar result is obtained by considering joint 4. For this joint clockwise reading gives the designation of the bar as IC, and ic in the stress diagram acts upward to the right, that is toward joint 4, since the bar is below this joint.

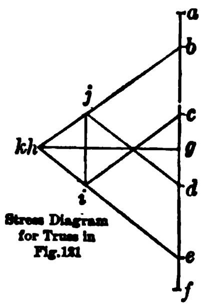


FIG. 122.

The example which follows represents clearly the application of this method and shows by the closure of the diagram that no error of importance has been made in the graphical work.

89. Application of Graphical Method of Joints. Fig. 121 shows the structure drawn to scale and with all the outer forces represented in direction and point of application. The force polygon is *abcdefga* in Fig. 122; this is a straight line, since all the forces are vertical. In it $ab = P_1$, $bc = P_2$, etc. The reactions R_L and R_R, represented by ga and fg, may be determined analytically, or graphically by methods given later.

The triangle of forces is first drawn for joint 1. The forces which act at this joint are R_L, P_1, the stress in bar BH, and the stress in bar HG, and these forces must be in equilibrium. Of these P_1 and R_L are known in magnitude and direction. Their resultant equals gb. The stresses in BH and HG are known

in direction but not in magnitude, hence there are but two unknown quantities at this joint, and it is evident that the value of these may be found by drawing a polygon of forces. The figure *gabhg* is such a polygon and is obtained by drawing from *b* a line parallel to *BH*, and from *g* a line parallel to *HG*. The line *bh* measured to the same scale as the force polygon gives the magnitude of the stress in the bar *BH*, and the line *hg* gives the magnitude of the stress in the bar *HG*. It remains to determine the character of these stresses. Considering joint 1, and reading around it in a clockwise direction, starting with *B*, gives *bh* acting downward to the left, that is from *B* toward *H*, thus showing compression. In the same manner the stress in *HG* is found to be tension, since it acts from *h* toward *g* or away from the joint. This method would not be correct had not the external forces been plotted by going around the figure in a clockwise direction, but it is evident that this being done the method is correct; since in order to have *ga*, *ab*, *bh* and *hg* in equilibrium the stresses in *BH* and *HG* must act as stated.

The next joint to be considered is joint 2, since there are now but two unknown forces acting there and they can therefore both be determined. To obtain them draw *ci* and *ih* in the force polygon parallel respectively to the corresponding bars in the truss; they will intersect at *i*. *ci* acts toward joint 2, and *ih* also acts toward this joint, hence compression occurs in both these bars. In a similar manner the stresses in the other bars may be determined.

90. Ambiguous Cases. The method of joints, graphical or analytical, is perfectly general and applicable to all trusses, but in order to apply it successfully to some types of trusses, it is necessary to choose the method of procedure with care. For example, in solving by the analytical method the truss shown in Fig. 123, it is not possible to consider the joints in succession beginning at the abutment, but after solving for the bars *BL*, *CM*, *LM*, *MN*, *LK* and *NK*, it is necessary to determine the stress in *PQ*. To do this apply the equations of equilibrium to joint at which P_4 is applied; using for axes the top chord and a line perpendicular to it. *po* may now be determined using as axes the bar *OR* and a line at right angles to that. It will then be possible to figure the stresses in the undetermined bars of that half of the truss.

In the graphical solution of this structure a similar difficulty also arises. After the stresses in bars BL, CM, LK, LM, MN and NK, and the corresponding bars in the other half of the truss have been determined, no joint exists at which only two unknown stresses act. To overcome this difficulty the following

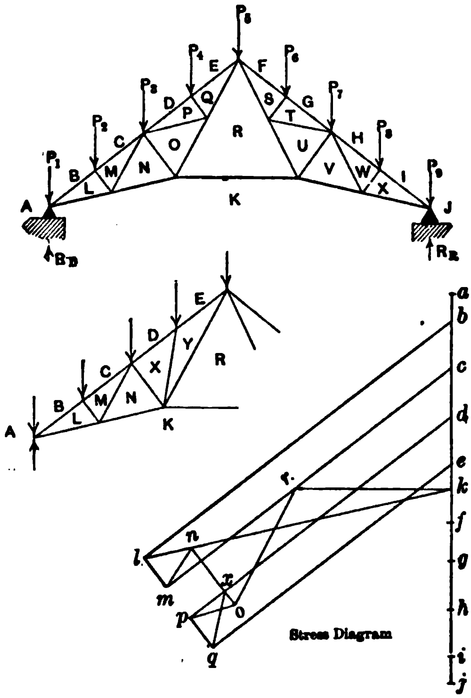


FIG. 123.

device may be employed: Consider the truss partially shown below the original truss in Fig. 123, in which bars PQ, QR, RO and OP of the original truss have been replaced by bars XY and YR. The truss is still determinate since one joint and two bars have been eliminated. Moreover, the stress in EY equals that in EQ since if the stress in EQ be computed by the analytical method of joints in the manner just described, but working from the right end of the truss, its value is clearly seen to be independent of any possible arrangement of bars on the left. The stress diagram for the new truss may now be continued and the point y located. This corresponds to q for the original truss, hence p is at the

intersection of qp and dp, and the remainder of the construction may be made without difficulty. The stress diagram for the left half of the truss is shown. That for the right half would be similar and is omitted.

This problem may also be solved by a combination of graphical and analytical methods, the stresses being determined analytically in such bars as are necessary and those values plotted in the diagram.

91. Analytical Method of Moments Described. This method of finding truss stresses is based upon the application of the equation, $\Sigma M = 0$. It is very useful for determining stresses in special bars of many trusses, but is not so general as the method of joints and is frequently inapplicable to many bars even in the simplest trusses. Like the method of joints, it is also a method of sections, the truss being considered as divided into two portions by a section and the equilibrium of one of these portions being considered. It can be used to determine the stress in a given bar when all the undetermined bars cut by the section except the one in question, or their prolongations, meet at a point, which point should be taken as the origin of moments.

92. Mode of Procedure. Method of Moments.

1. Assume the truss to be divided into two parts by an assumed section, which may be straight or curved. This section should cut the bar in which the stress is to be determined, and all the other bars cut by it should meet at a point which should not be on the bar in which it is desired to determine the stress, nor on its prolongation.

2. Apply the equation $\Sigma M = 0$, using the point of intersection described under 1 as the origin, and considering that portion of the truss giving the simpler equation. The equation must include the moment of all the outer forces acting on the portion of the truss under consideration, together with the moment of the unknown bar stress which should be assumed as tension. Clockwise moments should be considered as positive. The section is commonly taken as cutting but three bars, two of which meet at a point, while the third is the bar under consideration. It is sometimes simpler to deal with the moments of the components than with that of the forces themselves, particularly when the force may be resolved at a point such that the lever arms of one of the components is zero.

3. Solve the equation for the unknown stress. A positive result shows that the bar is in tension.

93. Method of Moments. Application. The application of this method is clearly illustrated by the following numerical example for the truss shown in Fig. 124.

Bar a. For this bar the XY section fulfils the required conditions; that is, it cuts three bars, two of which meet at a point, and the other is the bar a. If that portion of the truss to the left of the section be now considered, it is evident that it will be held in equilibrium by the outer forces and the stresses in bars a, b and c, and that the moment of the outer forces and the bar

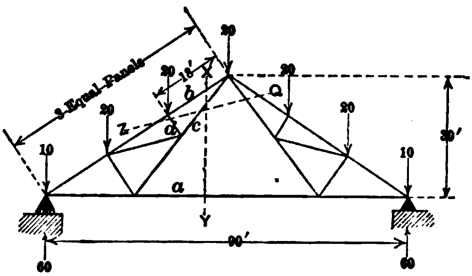


FIG. 124.

stresses about any point in the plane of the truss must equal zero. If moments be taken about the intersection of b and c, the moment of the stresses in these bars will be zero, hence the equation of moments will include only the known outer forces and the unknown stress in bar a, which can, in consequence, be readily computed. The equation will be as below, assuming stress in a as tension.

$$(60-10)45-20\times(15+30)-30S_a=0,$$

$$\therefore\ S_a=+45.$$

Bar b. The stress in bar b may be found by taking the origin of moments at the intersection of bar a and bar c, produced, using the same section as for bar a.

Bar d. In both of the cases considered the section XY was assumed to be vertical or nearly so; this, however, is not necessary, and a horizontal or inclined section may be used provided the bars cut by it fulfil the stated conditions. For example, the

stress in bar d may be computed by this method, using the section ZQ and taking moments about the apex of the truss of the forces above the section. The following equation is obtained for this case:

$$20\times 15+Sd\times 18=0 \quad \text{whence} \quad Sd=-20\times\frac{15}{18}=-16\tfrac{2}{3}.$$

It should be observed that the method of moments is inapplicable to the determination of the stresses in the web members of a parallel chord truss since in such trusses the origin of moments for the web member stress would be at infinity and the equations would be indeterminate.

94. Method of Shear. Described. This is another special method which can often be used to great advantage in the determination of the stresses in certain bars and particularly in diagonals of parallel chord trusses. In the truss shown in Fig. 125

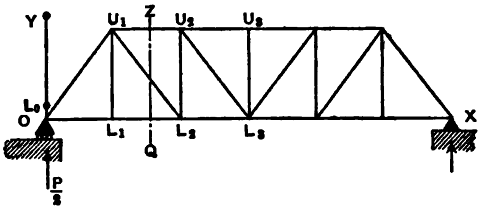


FIG. 125.

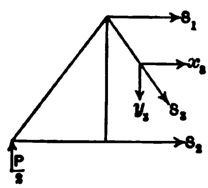


FIG. 126.

it is clear that if the stresses in all the bars are axial, the resultant forces perpendicular to the chords on either side of section ZQ must be carried entirely by the diagonal U_1L_2, and the application of the equation $\Sigma Y=0$ to all the forces on the portion of the truss to the left of the section gives at once an equation between the component in U_1L_2 parallel to the axis OY and the corresponding component of the outer forces. This is illustrated by Fig. 126, in which the application of $\Sigma Y=0$ gives the following equation:

$$\frac{P}{2}-y_3=0 \quad \text{whence} \quad y_3=\frac{P}{2}.$$

95. Mode of Procedure. Method of Shear. The treatment of the previous article shows the correctness of the following rules:

1. Divide the truss into two parts by a section passing through the bar in question. This section may cut any number of bars provided all are parallel except the one under consideration, but in general it should be so chosen as to cut not more than three bars.

2. Refer the forces to two axes, parallel and perpendicular respectively to the parallel bars cut by the section. Let the axis perpendicular to these bars be known as the Y axis. Determine the Y components of all the outer forces acting on that portion of the truss which has the fewer outer forces acting on it, and apply $\Sigma Y=0$. The equation should include the Y components of *all* the outer forces acting upon the portion of the truss selected, and the Y component of the unknown bar stress which should be assumed as tension.

3. Solve the equation thus obtained for the unknown Y component. A positive result shows that the stress in the bar is tension.

In most bridge trusses these conditions involve merely the application of $\Sigma V=0$ to the portion of the truss considered, i.e., the shear on the section equals the vertical component of the stress in the given diagonal, hence this method is ordinarily called the method of shear.

96. Method of Shear. Application. The following example clearly illustrates the application of this method to the determination of the stresses in the web members of the simple bridge truss, with horizontal chords and carrying a uniform dead and live load, shown in Fig. 127.

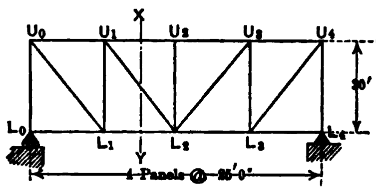


FIG. 127.

Let the dead load be taken as 1000 lbs. per foot all on the bottom chord, and the live load as 2000 lbs. per foot also on the bottom chord. The panel loads will then be 25,000 lbs. dead and 50,000 lbs. live, and the positive dead shear on the section XY will be 12,500 lbs. To get the maximum live shear on XY assume full

panel loads at panel points L_2 and L_3, and no load at panel point L_1. This gives a live shear in the panel of +37,500 lbs. The vertical component in the bar U_1L_2 will then be +12,500 lbs. dead and +37,500 lbs. live. With the vertical component known the actual stress can be easily computed. The forces acting upon that portion of the truss to the left of the section will be as shown in Fig. 128, from which it is readily seen that by assuming the stress in the diagonal to be tension and applying the equation $\Sigma V=0$, V_1 will be found to have a positive value.

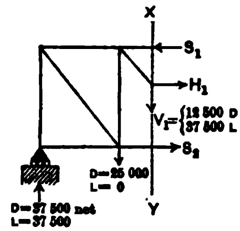


FIG. 128.

Had the truss been inclined instead of horizontal the proper course to pursue would have been to resolve the vertical forces into normal and tangential components, and apply $\Sigma Y=0$ to the normal forces.

97. General Rules for Determination of Truss Stresses. The student should note carefully that the three methods which have been explained, viz., the method of joints, the method of moments, and the method of shears, are all methods of sections, and that in their application it is always necessary to assume a section through the truss and write an equation of equilibrium between the outer forces acting upon the truss on one side of the section and the forces in the bars cut by the section. In computing a bar by analytical methods the first step is to determine the method to use. It should next be decided where to take the section and what portion of the truss to consider. Finally the proper equations should be applied between all the outer forces acting on the portion selected and the stresses in the bars cut.

A combination of the three methods which have been explained, joints, moments, and shears, enables us to compute readily the stress in any or all members of a statically determined truss. In order, however, to figure the stresses in the simplest manner, it may be necessary to study with considerable care some of the members, in order to determine which method should be adopted. In bridges, however, the forms of trusses which are in common use for simple spans are not numerous, and the best methods to adopt can be readily learned by the study of conventional types. For roof trusses the graphical method of

joints will usually be found most convenient though sometimes it may be desirable to supplement this method by computing the stress in certain bars by one of the other methods.

98. Counters. In pin trusses as commonly built in the United States the main diagonals are flat eye-bars which can carry little or no compression, and which are so placed in the truss as to be in tension under the dead load. Certain positions of the live load will, however, always tend to produce compression in some of the diagonals. This frequently overbalances the dead tension, especially when impact is added. Such is usually the case in panels near the centre of a railroad bridge truss where the dead stresses in the diagonals are small and the live stresses proportionally great. To prevent danger of collapse when this occurs it is necessary either to make the main diagonals of such a shape that they will carry compression, or else to relieve them by auxiliary diagonals, called counters, which are so placed that they will be brought into tension by that loading which would tend to put the main diagonal into compression. This latter method is the common practice, although in recent years recognition of the importance of rigidity as well as strength in railroad bridges has induced many engineers to use the former method, even at the sacrifice of simpler and less expensive details. For trusses such as the Howe truss, described in the following article, in which the main diagonals are compression members, but unsuited on account of end details to transmit tension, counters are needed to resist tension rather than compression.

With riveted trusses it is usually desirable to make all the web members of such a shape that they can carry both tension and compression and the question of counters does not arise.

To determine whether counters are needed, if it is considered desirable to use them, the live loads should be placed in the position consistent with maximum compression in each diagonal, or in the Howe truss with maximum tension, beginning with that in the panel nearest the centre, and proceeding toward the end, and the live stress computed. If this stress when combined with a reasonable allowance for impact equals or exceeds the dead stress of the opposite character in the bar, a counter is needed. It is wise to use a high allowance for impact in such a case, as the consequence of an increase in the live loads sufficient to overbalance the dead stress, would be more serious here than for a

bar where such an increase would tend merely to increase the unit stress in the member.

In trusses without counters it is necessary to make similar computations, since if reversal of stress occurs in a bar, it should be designed with a lower unit stress than would otherwise be adopted; at least if the reversal of stress occurs suddenly and frequently, as in a railroad bridge.

The reason for beginning at the centre and working toward the end in making these computations is to save labor. The ratio between the maximum live stress and the dead stress in the web members is always greater at the centre (that is for the ordinary end-supported truss) and grows less near the end. In consequence, after the panel, in which a counter has first been found unnecessary, is reached, no further investigation is required.

Illustrations of the computations to determine whether or not counters are needed are given in examples which follow. It may be helpful, however, to state here that in the ordinary parallel chord end-supported truss counters are needed wherever the negative live shear plus impact equals or exceeds the positive dead shear.

99. Types of Trusses. The forms of simple bridge trusses most frequently adopted are shown by Figs. 129 to 134 inclusive.

The Howe truss is usually built with chords, diagonals and end verticals of wood, and intermediate verticals of iron. Stresses

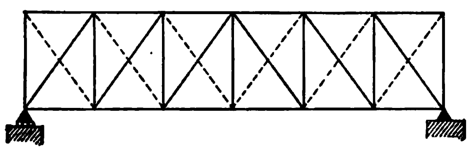


FIG. 129.—Howe Truss.

in diagonals will be compression, and in intermediate verticals tension. The diagonal members shown by the dotted lines are counters. Ordinarily in such trusses a counter is used in every panel though not needed to carry shear, its size being made one-half that of the main diagonal; that is, if two equal sticks are used for the main diagonal, the counter would be made one stick of the same size.

The Pratt truss is the most common type of bridge truss. It is usually built of steel, and has tension diagonals and compres-

sion verticals. The truss with end verticals shown in the upper portion of Fig. 130 is not commonly employed for through bridges since it is less economical of material than the other

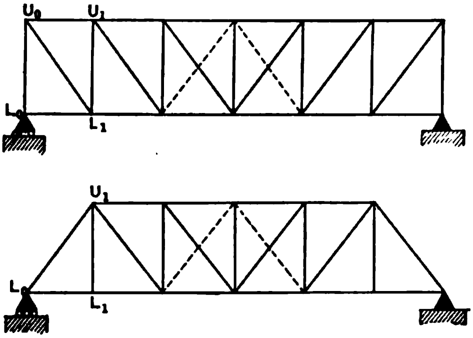


FIG. 130.—Pratt Truss.

form in which the compression members L_0U_0 and U_0U_1 and the tension member, U_0L_1, are replaced by the one compression member L_0U_1. Counters are shown dotted and may be required in more than two panels.

The Warren truss is very commonly adopted for riveted trusses of small span. No counters are used and the diagonals in panels where negative shear occurs are made compression members. It is evident, however, from the arrangement of the diagonals that every other one would, in any case, have to be a compression member to withstand the positive shear.

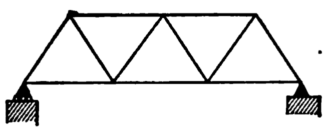


FIG. 131.—Warren Truss.

In all bridges, in order to obtain economy of material, it is essential that the ratio of depth of truss to length of span should be within certain limits, approximately $\frac{1}{5}$ to $\frac{1}{8}$, and that the diagonals should make angles of approximately 45° with the horizontal. To obtain both of these results it is clear that the panel length should vary with the span. As it is undesirable to use very long panels on account of the bending stresses produced in the chord bars by their own weight, if greater than 25 or 30 ft. in length, and because also of the increase in weight per foot of the stringers as their span increases, the panel length is seldom

made in excess of 35 ft., though in some spans of recent construction panel lengths greater than this have been used. In order to obtain panels of reasonable length in long spans, it is common

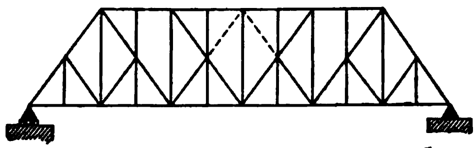


FIG. 132.—Sub-divided Pratt Truss commonly known as the Baltimore Truss.

to subdivide the truss by a secondary system, as indicated in Figs. 132 and 133, though such a sub-division causes, in some of the members, secondary stresses of considerable magnitude.

For very long spans it is usually more economical to make the truss deeper at the centre than at the ends. If the depth be

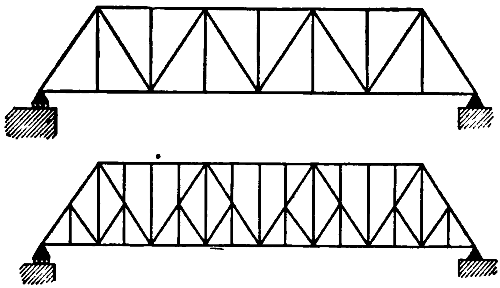


FIG. 133.—Sub-divided Warren Trusses.

increased in proportion to the increase in moment it is evident that the chord stresses would remain essentially constant throughout the entire length of the span, and that the chords would,

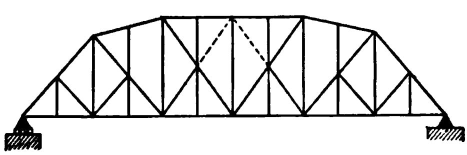


FIG. 134.—Sub-divided Pratt Truss with Inclined Top Chord commonly known as the Pettit Truss.

in consequence, be much lighter at the centre than if the end depth were to be continued throughout the span. The stresses in the diagonals would be increased in such a case, but the net result would be a saving of material, hence if minimum weight alone were to be the governing element, it would be desirable

to make all trusses of varying height. It is necessary, however, to consider also economy of labor. Since trusses of varying depth are more expensive of labor it is evident that they should be used only for structures in which the saving of weight balances or exceeds the increased cost of construction. This point is usually reached only in spans of considerable length, say 300 ft. and over, and the type of truss commonly used in such spans is shown by Fig. 134.

Roof trusses are necessarily made of many forms to suit the varying shapes of buildings. Figs. 135, 136 and 137 illustrate only a few of the more usual forms.

Fig. 135 shows a common type of roof truss which is built of steel or of wood with steel verticals. It has no special name but

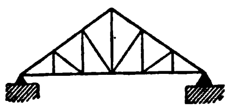


FIG. 135.

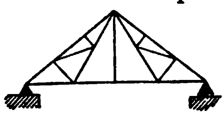


FIG. 136.

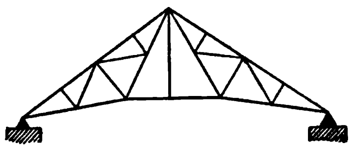


FIG. 137.
Common Types of Roof Trusses.

is of the Pratt truss type. Figs. 136 and 137 are Fink roof trusses.

100. Systems of Loading. In the computation of bridge truss stresses it is desirable to combine the various methods given in the preceding articles. Methods of doing this for the more common types of trusses and for simple loadings are clearly shown in the numerical examples which follow. As it is the writer's purpose in this chapter to lay particular emphasis upon truss action rather than upon the consideration of moving concentrated load systems which have already been treated, the live loading used in most of the examples is taken as a uniform load with a locomotive excess, that is, with a single concentrated live load which may be applied at any panel point. The magnitude of the locomotive excess load equals the difference between the maximum floor-beam load due to the actual locomotive and the floor-beam load due to the uniform load. The process of finding the maximum live stress in a member with this loading consists of computing

the maximum stress due to the uniform load, and adding to it the maximum stress due to the locomotive excess. The dead load is also treated as a uniform load, this being nearly correct for trusses of ordinary span. For trusses of great length or of unusual weight it is better to estimate the actual dead weight acting at each panel point.

It should be remarked that for parallel chord trusses the determination of the live stresses due to concentrated load systems involves merely the computation of the maximum moment at each panel point and the maximum shear in each panel. From the moments the chord stresses may be figured by the method of moments and from the shears the web stresses by the method of shears. If the student thoroughly understands truss action as illustrated in the examples which follow, and the method of using concentrated load systems, he should have no difficulty whatsoever in the computation of trusses under concentrated loads.

The fact that the locomotive excess method is used for the determination of truss stresses should not be considered as indicative of the writer's belief that such a method is sufficiently precise for actual use in design. It is used here merely because of its value in showing truss action without complicating the theory with unnecessary computations.

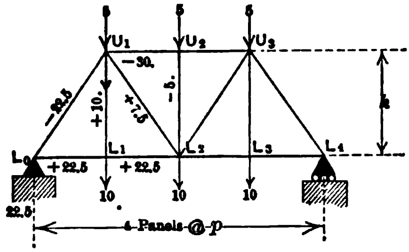
101. Index Stresses. For many bridge trusses the dead stresses and the stresses under full uniform live load can be most readily obtained by a special application of the method of joints, involving the use of so-called index stresses. The method of obtaining these index stresses, and a clear understanding of what they signify, may be gained from a study of the following example.

Let it be desired to determine the dead stresses in the simple truss shown in Fig. 138.

It is evident that the stress in U_2L_2 may be determined by the method of joints using the joint at U_2, and that its value is -5.

Since the truss and loads are *symmetrical* the stresses in U_1L_2 and L_2U_3 are equal, hence the vertical component in each may be found by considering joint L_2. Its value $= +\frac{1}{2}(5+10) = +7.5$. The stress in U_1L_1 is found to equal $+10$, using the joint at the bottom of the member, and the vertical component in $U_1L_0 = -(5+10+7.5) = -22.5$, considering the joint at U_1.

These vertical components of the web stresses are the web index stresses and may be written directly on the truss diagram and the dead stresses computed from them by the slide rule with great rapidity.



Loads shown are dead panel loads.

FIG. 138.

It should be noticed that the vertical component in L_0U_1 equals the left reaction (i.e., the net reaction, neglecting the panel load at L_0) and that a very good check upon the web stresses is thereby obtained.

Had the truss or the loads been *unsymmetrical* it would have been desirable to have started by writing first the vertical component in L_0U_1 and proceeding thence to the right end of the truss, checking there with the right reaction. For symmetrical structures symmetrically loaded it is better to begin at the centre working toward the end.

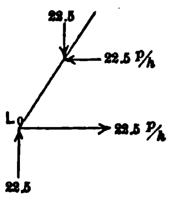


FIG. 139.

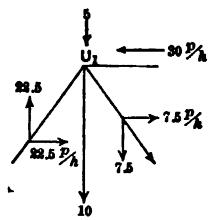


FIG. 140.

To obtain the chord stresses begin at L_0. The conditions necessary for equilibrium at that point are shown by Fig. 139. The actual stress in L_0L_1 is found to be $+22.5\frac{p}{h}$.

The condition at joint U_1 is shown by Fig. 140, and the stress in U_1U_2 is found to equal $30\frac{p}{h}$.

These numerical coefficients of the chord stresses are called the chord index stresses. For the truss in question it is evident that their determination requires merely the progressive addition, joint by joint, of the index stresses in the diagonals of the web system, and that this would be the case in all simple equal-panelled trusses of the Howe, Warren or Pratt types. For the subdivided trusses of the Baltimore or Petit type the effect of the secondary diagonals must be carefully considered, since the index stresses in these must sometimes be added and sometimes subtracted to obtain the chord index stresses.

It should be observed that in the case of a parallel-chord equal-paneled truss of height equal to panel length the chord index stresses equal the actual chord stresses, and that for other heights the latter vary inversely as the truss height.

For trusses in which the diagonals do not all have the same slope the web index stresses must all be reduced to a standard slope before writing the chord index stresses. The method of doing this is fully explained later in the article on trusses with non-parallel chords and will not be given here.

The chord index stress in the bar nearest the centre, or in the centre bar, if the truss has an uneven number of panels should be verified by comparing the actual stress as obtained from it with that obtained in the same bar by the method of moments, using the formula $\frac{1}{8}wL^2$ for this purpose if the load per foot is uniform. If the two results agree it is evident that not only the index stress in this bar will be checked, but also the index stresses in all the other members of the same chord. Moreover, the index stresses in the members of the other chord may be verified so easily by comparison with those in the chord already checked that no excuse need exist for errors.

This method is so advantageous, both from the standpoint of accuracy and rapidity, that it should invariably be used for simple bridge trusses. The numerous examples which follow illustrate it fully and should be carefully studied.

102. Computation of Stresses. Pratt Truss. *Uniform Load with Locomotive Excess.*

Problem. Determine the maximum stresses in all the members of the truss shown in Fig. 141 with the following loads:

Dead weight of bridge,

400 lbs. per ft. per truss, top chord	=10,000 lbs. per panel.	
1,000 " " " bottom "	=25,000 " "	

Uniform live load,

3,000 lbs, per ft. per truss, bottom chord = 75,000 lbs. per panel.
Locomotive excess,[1] " " = 40,000 "

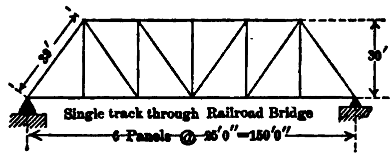


FIG. 141.

Index Stresses. These are shown for the dead loads in units of 1000 lbs. in Fig. 142, and can be written directly on the diagram, the necessary computations being done mentally.

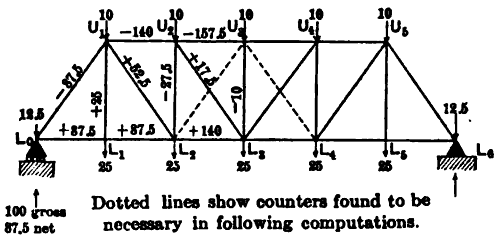


FIG. 142.—Dead Panel Loads and Index Stresses. Single-Track Through Pratt Truss Railroad Bridge.

gram, the necessary computations being done mentally. The vertical component in L_0U_1 equals the net left reaction as should be the case.

The stress in U_2U_3 as found from the index stress,

$$= 157.5 \times \frac{25}{30} = 131.25 \text{ thousands of lbs.}$$

[1] The locomotive excess is computed for Cooper's E_{60} loading (see Fig. 11). For this loading the maximum floor beam reaction for 25-ft. panels occurs with load (4) at the floor beam and equals 227,000 lbs. The corresponding reaction for the uniform live load = 6000 × 25 = 150,000 lbs. The total excess is therefore 227,000 − 150,000 = 77,000 lbs. The excess per truss = $\frac{77,000}{2}$ = 38,500, or, say, 40,000 lbs.

The stress in the same bar by the method of moments,

$$=\tfrac{1}{8}\cdot(1000+400)\frac{150^2}{30}=131{,}250 \text{ lbs.}$$

This agrees with the value found from the index stress and hence checks that stress and all the other index stresses involved in its determination.

The index stress in chord L_2L_3 plus the index stress in U_2L_3 equals the index stress in U_2U_3 as it should, hence the index stress in L_2L_3 is also correct.

Position of Loads for Maximum Live Stresses. Since the truss and loads are symmetrical the maximum stresses need be determined in the bars of one-half of the truss only. The maximum live stress in any one of the chord bars occurs for the loading, giving the maximum moment about some panel point, since the method of moments may be used for each of these bars, the origin in every ease being at a panel point. It follows from this that for maximum chord stresses the uniform live load should extend over the entire structure since only under this condition will the moment at a panel point be a maximum. In fact it may be stated as a general rule that *the maximum chord stresses due to a uniform load in any end-supported truss will occur only when the uniform load extends over the entire span.*

It is evident therefore that the maximum chord stresses due to the uniform live load may be obtained directly from the dead stresses by multiplying the latter by the ratio between the live panel load and the combined dead panel loads on top and bottom chords. If desired, however, the index stresses may be written for the uniform live load just as for the dead load and the actual stresses computed independently.

The maximum chord stresses due to the locomotive excess should be determined by the method of moments and will evidently, in this form of truss, occur with the excess load located in a vertical line through the panel point which is the origin of movements for the load in question. Its position for the various bars will be as follows:

Bars L_0L_1 and L_1L_2, E at L_1.

Bars U_1U_2 and L_2L_3, E at L_2.

Bar U_2U_3, E at L_3.

For the web stresses the uniform load and locomotive excess should generally be so placed as to give maximum shear in the

various panels. The only exception to this for the truss in question is that bar U_1L_1 has its maximum stress with a full panel load at L_1. It should be noticed that the stress in a vertical like U_2L_2 will be a maximum when the stress in the diagonal U_2L_3 is a maximum, since by the application of the method of joints it is evident that the stress in U_2L_2 equals the vertical component in U_2L_3. Were this condition not to exist, as would be the case if the live load should be distributed between the top and bottom chords, then the position of loads for maximum stress in U_2L_2 would have to be that which would give maximum shear in a diagonal section through U_1U_2 and L_2L_3. It should also be noticed that live stress in bar U_3L_3, if the load is applied to the bottom chord, occurs only when a counter is in action.

The position of loads for maximum stresses in the various bars will now be given, it being understood that the shear due to uniform live load will be treated by the approximate method hitherto used.

Bar L_0U_1, uniform live load over entire structure, E at L_1.

Bar U_1L_1, full uniform live panel load at L_1, E at L_1.

Bar U_1L_2, uniform load from right up to and including L_2, E at L_2.

Bars U_2L_3 and U_2L_2, uniform load from right up to and including L_3, E at L_3.

Bars U_3L_3 and U_3L_4, (counter) uniform load from right up to and including L_4, E at L_4.

Maximum Stresses. The actual stresses may now be computed. These are given with all the necessary computations in the table on following page.

It should be noted that it is simpler to determine the vertical components in all the diagonal bars before determining the actual stresses, particularly if the slide rule is used. In addition to the bar stresses the maximum truss reactions must be determined. These occur for full loading, but their values depend upon whether an end floor beam is used. If an end floor beam is not used the reaction equals the maximum shear in the end panel, the expression for the value of which has already been found in determining the stress in L_0U_1. If an end floor beam is used the locomotive excess should be placed at L_0, hence its value plus that of a half panel load of the uniform load should be added to the maximum shear in the end panel.

MAXIMUM STRESSES IN UNITS OF 1000 LBS.

Bar.	Index Stress.	Multiplier.	Dead Stress.	Live Stress Due to Uniform Load.	Live Stress Due to Locomotive Excess.	Total Live Stress.
L_0L_1	+ 87.5	25/30	+ 72.9	$+\ 72.9\times\frac{75}{35}=+156.2$	$+\frac{5}{6}\times40\times\frac{25}{30}=+27.8$	+184.0
L_1L_2	+ 87.5	25/30	+ 72.9	$+\ 72.9\times\frac{75}{35}=+156.2$	$+\frac{5}{6}\times40\times\frac{25}{30}=+27.8$	+184.0
U_1U_2	−140.0	25/30	−116.7	$-116.7\times\frac{75}{35}=-250.0$	$-\frac{4}{6}\times40\times\frac{50}{30}=-44.4$	−294.4
L_2L_3	+140.0	25/30	+116.7	$+116.7\times\frac{75}{35}=+250.0$	$+\frac{4}{6}\times40\times\frac{50}{30}=+44.4$	+294.4
U_2U_3	−157.5	25/30	−131.2	$-131.2\times\frac{75}{35}=-281.1$	$-\frac{3}{6}\times40\times\frac{75}{30}=-50.0$	−331.1
L_0U_1	− 87.5	39/30	−113.8	$-113.8\times\frac{75}{35}=-243.9$	$-\frac{5}{6}\times40\times\frac{39}{30}=-43.3$	−287.2
U_1L_2	+ 52.5	39/30	+ 68.3	$+\ \frac{10}{6}75\times\frac{39}{30}=+162.5$	$+\frac{4}{6}\times40\times\frac{39}{30}=+34.7$	+197.2
U_2L_3	+ 17.5	39/30	+ 22.8	$+\ \frac{6}{6}75\times\frac{39}{30}=+\ 97.5$	$+\frac{3}{6}\times40\times\frac{39}{30}=+26.0$	+123.5
U_1L_1	+ 25.0	1.0	+ 25.0	$+\ 75.0\times1.0=+\ 75.0$	$=+40.0$	+115.0
U_2L_2	− 27.5	1.0	− 27.5	$-\ \frac{6}{6}75\times1.0=-\ 75.0$	$-\frac{3}{6}\times40=-20.0$	− 95.0
U_3L_3	− 10.0	1.0	− 10.0	$-\ \frac{3}{6}75\times1.0=-\ 37.5$	$-\frac{2}{6}\times40=-13.3$	− 50.8
U_3L_4	Counter. Not in action under dead load only.			$+\ \frac{3}{6}75\times\frac{39}{30}=+\ 48.8$	$+\frac{2}{6}\times40\times\frac{39}{30}=+17.3$	+ 66.1

103. Computation of Stresses. Warren Truss. *Uniform Load with Locomotive Excess.*

Problem. Determine the maximum stresses of both kinds for all the bars of the truss shown in Fig. 143 with the following loads:

Dead weight of bridge,

600 lbs. per ft. per truss,	top chord	= 9,000 lbs. per panel.
200 " "	bottom chord	= 3,000 " panel.

Uniform live load,

2000 " "	top chord	=30,000 " panel.

Locomotive excess, " " =25,000 lbs.

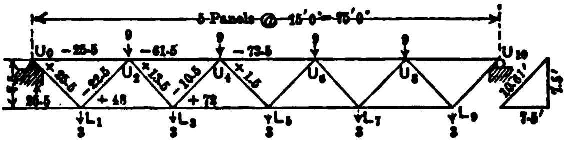


FIG. 143.—Dead Panel Loads and Index Stresses. Single-Track Deck Warren Truss Railroad Bridge.

The bottom panel loads at L_1 and L_9 would really be somewhat less than shown, but they are taken the same as the other bottom panel loads for convenience.

Index Stresses. These and the panel loads are shown in Fig. 143 in units of 1000 lbs. The net reaction at left end evidently equals 25.5 which checks the index stress in U_0L_1.

To check the index stress in the centre member of the top chord moments should be taken about L_5. Since this is not a panel point for both chords this moment does not equal $\frac{1}{8}pL^2$, but may be found from the moment of the reaction minus the moment of the panel loads. By this method the stress in the centre top chord bar equals

$$(25.5\times 2\tfrac{1}{2} - 3\times 3 - 9\times 2)\,\frac{15}{7.5} = 73.5.$$

Since the diagonals make an angle of 45° with the horizontal the chord index stress equals the actual stress and is therefore correct.

The index stress in the centre member of the bottom chord plus the index stress in $U_4L_5 = 73.5 =$ the index stress in the centre member of the top chord, and is therefore correct.

For this truss the live stresses cannot be computed from the dead index stresses since the bottom chord joints are not directly under the top chord joints; as the chord stresses for the uniform live load have maximum values for full loading they can, however, be determined by the method of index stresses, and Fig. 144 shows these stresses for a full uniform live load.

The moment at U_4 for this case equals $\left(\frac{24}{25}\right)\cdot(\frac{1}{8}\cdot wL^2)$, using

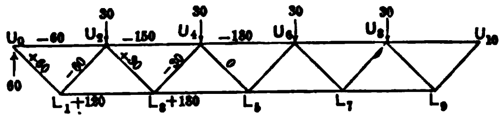


FIG. 144.—Panel Loads and Index Stresses. Full Live Load.

the method of Art. 43, hence the tension in bar L_3L_5 in thousands of pounds

$$=\frac{24}{25}\cdot\frac{1}{8}\cdot 2\cdot\frac{75\times 75}{7.5}=180.$$

This equals the index stress in this member and also in U_4U_6 as should be the case since the index stress in diagonal $U_4L_5=0$.

Position of Loads for Maximum Live Stresses:

U_0L_1 and L_1U_2, full uniform load with E at U_2.

U_2L_3 and L_3U_4, uniform load from right up to and including U_4, E at U_4.

U_4L_5, uniform load from right up to and including U_6, E at U_6.

L_5U_6 (= maximum compression in U_4L_5), uniform load from right up to and including U_6, E at U_6.

U_6L_7 (= maximum tension in U_4L_3), full panel load and locomotive excess at U_8.

U_0U_2 and L_1L_3, full uniform load, E at U_2.

U_2U_4 and L_3L_5, full uniform load, E at U_4.

U_4U_6, full uniform load, E at U_4 or U_6.

Maximum Stresses. These may now be computed and are given in thousand-pound units in the following table in which all the necessary computations are shown:

MAXIMUM STRESSES—WARREN TRUSS.

Bar.	Dead Stress in Units of 1000 lbs.	Max. Live Stress in Units of 1000 lbs.
U_0U_2	-25.5	$-\left(60+\frac{4}{5}25\right) = -\ 80.0$
U_2U_4	-61.5	$-\left(150+\frac{3}{5}25\times3\right) = -195.0$
U_4U_6	-73.5	$-\left(180+\frac{2}{5}25\times5\right) = -230.0$
L_1L_3	$+48.0$	$+\left(120+\frac{4}{5}25\times2\right) = +160.0$
L_3L_5	$+72.0$	$+\left(180+\frac{3}{5}25\times4\right) = +240.0$
U_0L_1	$+25.5\times\frac{10.61}{7.5} = +36.1$	$+\left(60+\frac{4}{5}25\right)\frac{10.61}{7.5} = +113.1$
L_1U_2	$-22.5\times\frac{10.61}{7.5} = -31.8$	$-\left(60+\frac{4}{5}25\right)\frac{10.61}{7.5} = -113.1$
U_2L_3	$+13.5\times\frac{10.61}{7.5} = +19.1$	$+\left(\frac{6}{5}30+\frac{3}{5}25\right)\frac{10.61}{7.5} = +\ 72.1$
L_3U_4	$-10.5\times\frac{10.61}{7.5} = -14.8$	$-\left(\frac{6}{5}30+\frac{3}{5}25\right)\frac{10.61}{7.5} = -\ 72.1$
U_4L_5	$+\ 1.5\times\frac{10.61}{7.5} = +\ 2.1$	$+\left(\frac{3}{5}30+\frac{2}{5}25\right)\frac{10.61}{7.5} = +\ 39.6$
* L_5U_6	Same as U_4L_5 $= +\ 2.1$	$-\left(\frac{3}{5}30+\frac{2}{5}25\right)\frac{10.61}{7.5} = -\ 39.6$
* U_6L_7	Same as U_4L_3 $= -14.8$	$+\ \frac{1}{5}(30+25)\frac{10.61}{7.5} = +\ 15.6$

* In this truss no counters are used, hence it is necessary to compute the maximum stresses of both kinds in all diagonals in which the live stress may tend to reverse the dead stress. This is easily done in the manner shown above.

104. Computation of Stresses, Subdivided Warren Truss. *Uniform Load with Locomotive Excess.*

Problem. Determine the maximum stress of both kinds in all the bars of the truss shown in Fig. 145 with the following loads:

Dead weight of bridge,

1000 lbs. per ft. per truss, top chord = 25,000 lbs. per panel

500 " " " " " bottom " = 12,500 " " "

Uniform live load,

2,000 lbs. per ft. per truss, top chord = 50,000 lbs. per panel

Locomotive excess, = 30,000 "

Index Stresses. These are shown in Fig. 145. Their computation involves no difficulty.

To check the index stress in U_4U_5 use the method of moments as follows:

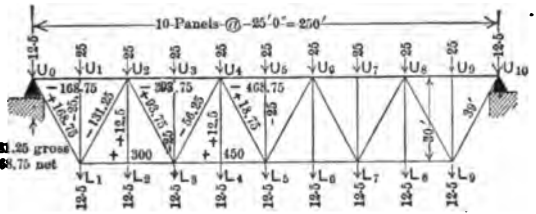


FIG. 145.—Dead Panel Loads and Index Stresses. Single-Track Deck Subdivided Warren Truss Railroad Bridge.

$$\text{Stress in } U_4U_5 = \tfrac{1}{8} \times 1500 \cdot \frac{250 \cdot 250}{30} = 390{,}625 \text{ lbs.}$$

From the index stress the actual stress in this bar $= 468{,}750 \times \dfrac{25}{30}$ $= 390{,}625$, hence index stresses are correct.

Position of Loads for Maximum Live Stresses. For the chords the uniform live load should extend over the entire span and the stresses due to it may be computed directly from the dead stresses. The locomotive excess should be placed as follows:

Bars U_0U_1 and U_1U_2,	E at U_1.	Bars U_2U_3 and U_3U_4.	E at U_3.
Bars L_1L_2 and L_2L_3,	E at U_2.	Bars L_3L_4 and L_4L_5.	E at U_4.
		Bar U_4U_5,	E at U_5.

For the diagonals the uniform load and locomotive excess should be placed to give maximum shear in the different panels, i.e., full uniform live panel loads to the right of the panel containing the bar in question, and the locomotive excess at the nearest panel point to the right. For the verticals it is evident that the maximum stress in all odd numbered bars like U_3L_3 will occur with full live panel load and locomotive excess at top panel point, while the even numbered verticals will have no live stress.

Maximum Stresses. All necessary computations for these, together with the final values, are given in the following table in units of 1000 lbs:

MAXIMUM STRESSES—SUBDIVIDED WARREN TRUSS

Bars.	Index Stress.	Multiplier.	Dead Stress. 1000 lb. Units.	Live Stress Due to Uniform Load. 1000 lb. Units.
U_0U_1, U_1U_2	-168.75	$\frac{25}{30}$	-140.6	$-140.6\times\frac{50}{37.5}=-187.5$
U_2U_3, U_3U_4	-393.75	$\frac{25}{30}$	-328.1	$-328.1\times\frac{50}{37.5}=-437.5$
U_4U_5	-468.75	$\frac{25}{30}$	-390.6	$-390.6\times\frac{50}{37.5}=-520.8$
L_1L_2, L_2L_3	$+300.00$	$\frac{25}{30}$	$+250.0$	$+250.0\times\frac{50}{37.5}=+333.3$
L_3L_4, L_4L_5	$+450$	$\frac{25}{30}$	$+375.0$	$+375.0\times\frac{50}{37.5}=+500.0$
U_0L_1	$+168.75$	$\frac{39}{30}$	$+219.4$	$+219.4\times\frac{50}{37.5}=+292.5$
L_1U_2	-131.25	$\frac{39}{30}$	-170.6	$-50\times\frac{36}{10}\times\frac{39}{30}=-234.0$
U_2L_3	$+\ 93.75$	$\frac{39}{30}$	$+121.9$	$+50\times\frac{28}{10}\times\frac{39}{30}=+182.0$
L_3U_4	$-\ 56.25$	$\frac{39}{30}$	$-\ 73.1$	$-50\times\frac{21}{10}\times\frac{39}{30}=-136.5$
U_4L_5	$+\ 18.75$	$\frac{39}{30}$	$+\ 24.4$	$+50\times\frac{15}{10}\times\frac{39}{30}=+\ 97.5$
* L_5U_6	$+\ 18.75$	$\frac{39}{30}$	$+\ 24.4$	$-50\times\frac{10}{10}\times\frac{39}{30}=-\ 65.0$
* U_6L_7	$-\ 56.25$	$\frac{39}{30}$	$-\ 73.1$	$+50\times\frac{6}{10}\times\frac{39}{30}=+\ 39.0$
* L_7U_8	$+\ 93.75$	$\frac{39}{30}$	$+121.9$	$-50\times\frac{3}{10}\times\frac{39}{30}=-\ 19.5$
U_1L_1, U_3L_3, U_5L_5	$-\ 25.0$	1.0	$-\ 25.0$	$=-\ 50.0$
U_2L_2, U_4L_4	$+\ 12.5$	1.0	$+\ 12.5$	0.0

* The live stresses in these bars are maximum stresses of the opposite character to those occurring in the corresponding bars in the other half of the truss.

(*Table continued on next page.*)

105. Computation of Stresses. Bridge Trusses with Non-parallel Chords. *Uniform Load with Locomotive Excess.* To compute the stresses in such trusses it is necessary to modify somewhat the procedure adopted in the simple parallel chord trusses hitherto treated. This is due to the fact that the web stresses can no longer be directly determined by the method of shear, owing to the influence of the inclined top chord. Although the modification is in mode of procedure rather than in principle,

MAXIMUM STRESSES—SUBDIVIDED WARREN TRUSS

	Live Stress Due to Locomotive Excess. 1000 lb. Units.	Total Live Stress. 1000 lb. Units.
U_0U_1, U_1U_2	$-\frac{9}{10}\times 30\times\frac{25}{30}=-22.5$	-210.0
U_2U_3, U_3U_4	$-\frac{7}{10}\times 30\times\frac{75}{30}=-52.5$	-490.0
U_4U_5	$-\frac{5}{10}\times 30\times\frac{125}{30}=-62.5$	-583.3
L_1L_2, L_2L_3	$+\frac{8}{10}\times 30\times\frac{50}{30}=+40.0$	$+373.3$
L_3L_4, L_4L_5	$+\frac{6}{10}\times 30\times\frac{100}{30}=+60.0$	$+560.0$
U_0L_1	$+\frac{9}{10}\times 30\times\frac{39}{30}=+35.1$	$+327.6$
L_1U_2	$-\frac{8}{10}\times 30\times\frac{39}{30}=-31.2$	-265.2
U_2L_3	$+\frac{7}{10}\times 30\times\frac{39}{30}=+27.3$	$+209.3$
L_3U_4	$-\frac{6}{10}\times 30\times\frac{39}{30}=-23.4$	-159.9
U_4L_5	$+\frac{5}{10}\times 30\times\frac{39}{30}=+19.5$	$+117.0$
$*L_5U_6$	$-\frac{4}{10}\times 30\times\frac{39}{30}=-15.6$	$-\ 80.6$
$*U_6L_7$	$+\frac{3}{10}\times 30\times\frac{39}{30}=+11.7$	$+\ 50.7$
$*L_7U_8$	$-\frac{2}{10}\times 30\times\frac{39}{30}=-\ 7.8$	$-\ 27.3$
U_1L_1, U_3L_3, U_5L_5	-30.0	$-\ 80.0$
U_2L_2, U_4L_4	0.0	0.0

it seems desirable to illustrate the necessary computations for such a truss, hence the following example is given.

Problem. Compute the maximum stresses for the truss shown in Fig. 146 with the following loads:

Dead weight of bridge,

600 lbs. per ft. per truss, top chord = 15,000 lbs. per panel.
1200 " " " " " bottom " = 30,000 " " "

Uniform live load,

3000 lbs. per ft. per truss, bottom chord = 75,000 " " "

Locomotive excess, = 40,000 "

The determination of index stresses for this truss requires some explanation. The inclination of top chord members adds vertical forces at joints U_1, U_2 and U_3, hence the vertical components in the inclined chord members must be determined before the index stresses for bars meeting at these joints can be written. In the trusses previously considered all the diagonals had the same slope, and multiplication of the chord index stresses by the ratio of horizontal to vertical projection of the diagonal gave actual stresses. It is obvious that in order to follow this same method in the truss under consideration, some modification must be adopted. The simplest method in this case is to correct the index stresses in bars U_2L_3 and U_3L_4, before writing chord index stresses. The best method of accomplishing this is to multiply

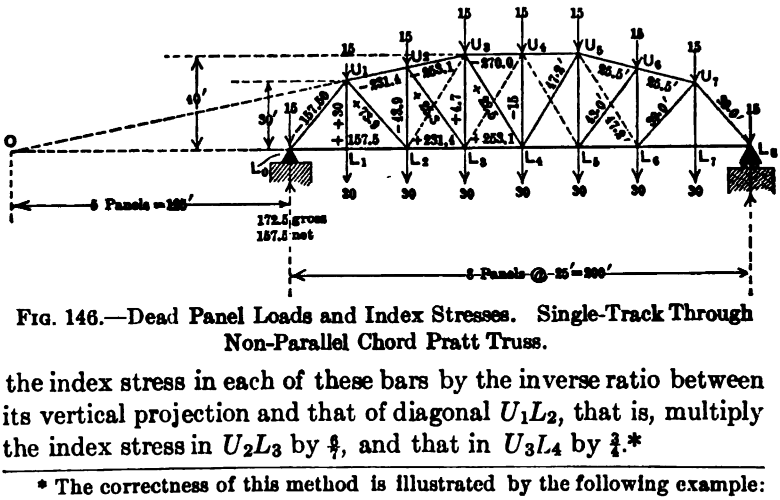


FIG. 146.—Dead Panel Loads and Index Stresses. Single-Track Through Non-Parallel Chord Pratt Truss.

the index stress in each of these bars by the inverse ratio between its vertical projection and that of diagonal U_1L_2, that is, multiply the index stress in U_2L_3 by $\frac{6}{7}$, and that in U_3L_4 by $\frac{3}{4}$.*

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* The correctness of this method is illustrated by the following example: Let V_1, V_2, and V_3 be the vertical components in the diagonals of truss shown in Fig. 147, and H_1, H_2 and H_3 the stresses in the bottom chord Evidently

$$H_1 = V_1 \times \frac{p}{h}, \quad H_2 = (V_1 + V_2)\frac{p}{h},$$

and $$H_3 = (V_1 + V_2)\frac{p}{h} + V_3 \times \frac{p}{h_1}$$

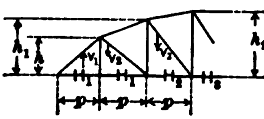


FIG. 147.

$$= (V_1 + V_2)\frac{p}{h} + V_3\left(\frac{p}{h_1}\right)\frac{h}{h}$$

$$= (V_1 + V_2)\frac{p}{h} + \left(V_3 \times \frac{h}{h_1}\right)\frac{p}{h}$$

$$= \left(V_1 + V_2 + V_3 \times \frac{h}{h_1}\right)\frac{p}{h}.$$

For trusses in which the height is constant, but the panel length variable, the same method may be applied with the panel lengths substituted for the heights.

The effect of this is to reduce the chord index stresses to the values they would have if all the diagonals had the same slope as U_1L_2.

The computation of the vertical components due to dead load in the inclined top chord members follows:

$$\text{V.C. Bar } U_1U_2 - \text{Dead Load } \left(157.5\times\frac{50}{35}-45\times\frac{25}{35}\right)\frac{10}{50}=38.6$$

$$\text{V.C. Bar } U_2U_3 - \text{Dead Load } \left(157.5\times\frac{75}{40}-45\times\frac{75}{40}\right)\frac{10}{50}=42.2.$$

With these known, the vertical components in the web members may be written at once, beginning at the centre in the usual manner and obtaining a check at the end where the vertical component in the inclined end-post L_0U_1 is found to equal the net dead reaction. It will be noticed that the effect of the vertical component in U_2U_3 is to cause tension in bar U_3L_3. In a parallel chord truss this member, with the other verticals, would always be in compression under dead load.

The corrected values of the diagonal index stresses which are to be used to determine the chord index stresses are as follows:

$$\text{Bar } U_2L_3=25.3\times\tfrac{6}{7}=21.7,$$
$$\text{Bar } U_3L_4=22.5\times\tfrac{3}{4}=16.9.$$

These values should be substituted for the actual diagonal index stresses in determining the chord index stresses. For example, the index stress in Bar $L_3L_4=231.4+21.7=253.1$, and that in $U_3U_4=253.1+16.9=270.0$.

To obtain a final check of these index stresses, the top chord dead stress, as computed by the method of moments, should be compared with the value as obtained from the index stresses.

Dead stress in U_3U_4 by method of moments

$$=\left(\frac{1}{8}\cdot1800\cdot200\cdot200\right)\frac{1}{40}=225{,}000 \text{ lbs.}$$

Dead stress in U_3U_4 from index stress

$$=270\times\frac{25}{30}=225 \text{ thousands of lbs.}$$

Position of Loads for Maximum Live Web Stresses. The dead stresses and chord and inclined end-post stresses due to uniform live load may be determined directly from the index stresses, and will be given later. To determine the *position* of the live loads for maximum web stresses, the method of shear previously used should be replaced by the method of moments. In determining the *actual* stress, once the position of loads is known, the method of shear may be used, provided the shear be corrected by the amount of the vertical component in the top chord, or the method of moments may be used directly. The individual bars will now be considered.

Bar U_1L_1: Maximum live stress with full load at L_1, E at L_1.

Bar U_1L_2: Place load to give maximum counter-clockwise moment about the intersection O of the inclined top chord, prolonged, and the bottom chord. Evidently a load at L_1 will cause a clockwise moment about this origin, since the moment of the reaction due to a load at L_1 will be less than the moment of the load itself as its lever arm and magnitude are both less, hence this point should not be loaded, but all other panel points from the right up to and including L_2 should be loaded (note that this conclusion would not necessarily be correct for a concentrated load system). The locomotive excess should be placed at L_2.

Bars U_2L_2 and U_2L_3: In this case load from right up to and including L_3, with E at L_3, since this condition produces the maximum counter-clockwise moment about O.

Bars U_3L_3 and U_3L_4: Load from right up to and including L_4 with E at L_4,

Bars U_4L_4, and U_4L_5 (counter): Load from right up to and including L_5 with E at L_5.

Bar U_5L_6 (counter): Load from right up to and including L_6 with E at L_6.

The tables which follow show the maximum stresses in all bars with all necessary computations.

MAXIMUM CHORD AND INCLINED END-POST STRESSES, IN UNITS OF 1000 LBS.

This table shows all necessary computations

Bar.	Index Stresses.	Dead Stresses.	Live Stresses.		Total Maximum Live Stress.
			Uniform Load.	Locomotive Excess.	
L_0U_1	157.5	$157.5\times\frac{39}{30} = -204.8$	$204.8\times\frac{75}{45} = -341.4$	$\frac{7}{8}40\times\frac{39}{30} = -45.5$	-386.9
U_1U_2	231.4	$231.4\times\frac{25}{30}\times\frac{25.5}{25} = -196.7$	$196.7\times\frac{75}{45} = -327.8$	$\frac{3}{4}40\times\frac{50}{35}\times\frac{25.5}{25} = -43.7$	-371.5
U_2U_3	253.1	$253.1\times\frac{25}{30}\times\frac{25.5}{25} = -215.1$	$215.1\times\frac{75}{45} = -358.6$	$\frac{5}{8}40\times\frac{75}{40}\times\frac{25.5}{25} = -47.8$	-406.4
U_3U_4	270.0	$270\times\frac{25}{30} = -225.0$	$225\times\frac{75}{45} = -375.0$	$\frac{1}{2}40\times\frac{100}{40} = -50.0$	-425.0
L_0L_1 L_1L_2	157.5	$157.5\times\frac{25}{30} = +131.2$	$131.2\times\frac{75}{45} = +218.6$	$\frac{7}{8}40\times\frac{25}{30} = +29.2$	$+247.8$
L_2L_3	231.4	$231.4\times\frac{25}{30} = +192.8$	$192.8\times\frac{75}{45} = +321.4$	$\frac{3}{4}40\times\frac{50}{35} = +42.9$	$+364.3$
L_3L_4	253.1	$253.1\times\frac{25}{30} = +210.9$	$210.9\times\frac{75}{45} = +351.5$	$\frac{5}{8}40\times\frac{75}{40} = +46.9$	$+398.4$

MAXIMUM LIVE WEB STRESSES, IN UNITS OF 1000 LBS.

This table shows all necessary computations

Bar.	Reaction = shear.	Components, Top Chord Stress Loading giving Maximum Stress in Bar. Horizontal.		Vertical.		Vertical Component Maximum Stress.	Maximum Live Stress.
U_1L_1							$+115.0$
U_1L_2	$\frac{21}{8}75+\frac{3}{4}40=226.9$	Bar U_1U_2	$226.9\times\frac{50}{35}=324.1$	U_1U_2	64.8	$226.9-64.8=162.1$	$+162.1\times\frac{39}{30}=+210.7$
U_2L_2	$\frac{15}{8}75+\frac{5}{8}40=165.6$	Bar U_1U_2	$165.6\times\frac{50}{35}=236.6$	U_1U_2	47.3	$165.6-47.3=118.3$	-118.3
U_2L_3	Same as $U_2L_2=165.6$	Bar U_2U_3	$165.6\times\frac{75}{40}=310.5$	U_2U_3	62.1	$165.6-62.1=103.5$	$+103.5\times\frac{43}{35}=+127.2$
U_3L_3	$\frac{10}{8}75+\frac{40}{2}=113.8$	Bar U_2U_3	$113.8\times\frac{75}{40}=213.3$	U_2U_3	42.7	$113.8-42.7=71.1$	-71.1
U_3L_4	Same as $U_3L_3=113.8$					113.8	$+113.8\times\frac{47.2}{40}=+134.3$
U_4L_4	$\frac{6}{8}75+\frac{3}{8}40=71.3$					71.3	-71.3
U_4L_5 / U_4L_3	Same as $U_4L_4=71.3$					71.3	$+71.3\times\frac{47.2}{40}=+84.1$
U_5L_6 / U_3L_2	$\frac{3}{8}75+\frac{1}{4}40=38.1$	Bar U_5U_6	$38.1\times\frac{150}{35}=163.3$	U_5U_6	32.7	$38.1+32.7=70.8$	$+70.8\times\frac{47.2}{40}=+83.5$

DEAD WEB STRESSES IN UNITS OF 1000 LBS.

This table shows all necessary computations.

Bar.	Index. Stresses.	Dead Stresses.	Bar.	Dead Stress.
U_1L_2	+73.9	$73.9\times\frac{39}{30}=+96.1$	U_1L_1	+30.0
U_2L_3	+25.3	$25.3\times\frac{43}{35}=+31.1$	U_2L_2	−43.9
U_3L_4	+22.5	$22.5\times\frac{47.2}{40}=+26.6$	U_3L_3	+ 4.7
$\left.\begin{matrix}U_4L_5\\U_4L_3\end{matrix}\right\}$	−22.5*	$22.5\times\frac{47.2}{40}=-26.6$	U_4L_4 †	−15.0
$\left.\begin{matrix}U_5L_6\\U_3L_2\end{matrix}\right\}$	−28.9‡	$28.9\times\frac{47.2}{40}=-34.1$		

* With live load placed to produce maximum stress in U_4L_4 the main diagonal, U_5L_4 will be thrown out of action and dead shear in panel 4–5 will be carried by the counter U_4L_5. This would tend to produce compression in this bar the vertical component of which is 22.5, but this is balanced by some of the live stress, hence the bar does not actually carry compression, as its sign would seem to indicate. A similar condition exists with bar U_5L_6.

† Owing to the counter action, the dead load when truss is loaded to produce maximum live compression in bar U_4L_4 tends to cause in the bar a tension of $22.5-15.0=7.5$. This value should be combined with the live stress to obtain maximum stress.

‡ $+28.9=157.5-90-\left(\frac{157.5\times 50-45\times 25}{35}\right)\frac{10}{50}$.

For convenience it is common to write the maximum stresses in a diagram called the stress diagram and Fig. 148 is given to illustrate such a diagram.

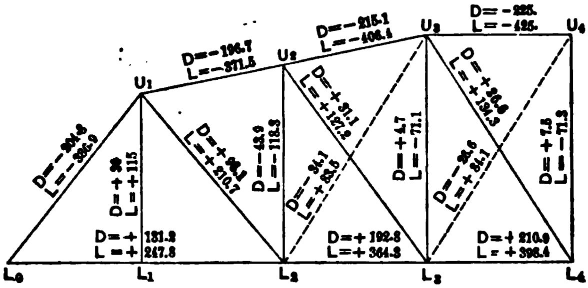


FIG. 148.—Stress Diagram, Maximum Live and Dead Stresses. Uniform Live Load with Locomotive Excess.

106. Computation of Stresses. Bridge Trusses with Non-Parallel Chords. *Concentrated Load Systems.* In the parallel chord trusses hitherto studied, the stresses due to concentrated load systems were not determined, since this would have involved merely the substitution of maximum shears and moments caused by such loads for the maximum shears and moments due to

uniform loads, and the method of determining such shears and moments was discussed with sufficient thoroughness in Chapter III. For non-parallel chord trusses, however, the conditions differ sufficiently to warrant special consideration; in consequence, the computations for the maximum live stresses due to Cooper's E_{40} loading will be given for the truss shown by Fig. 146. The computations for all the bars will be given, to make the solution complete, although in a number of cases it involves nothing more than the determination of maximum moment or shear. The moment diagram of Fig. 75 is used in all computations, and units are in thousands of pounds.

Position of Loads for Maximum Stresses: (Panel length $= p$)

Bars L_0U_1, L_0L_1 and L_1L_2: Maximum stress occurs when shear in panel $0-1$ is a maximum.

Try load (3) at L_1 and move up load (4). $(284+2\times 79)\frac{5}{8p}+\delta > 50\times\frac{5}{p}$.

Try load (4) at L_1 and move up load (5). $(284+2\times 84)\frac{5}{8p}+\delta < 70\times\frac{5}{p}$.

$\therefore$ Load (4) at L_1 gives maximum.

Bar U_1L_1: Maximum stress occurs with the loads placed in the position giving maximum moment at the centre of a beam 50 ft. long. See Article 51.

Try load (3)	$50 < 66$	$\therefore$ not a maximum.
Try load (4)	$59 < 70$	$\therefore$ a maximum.
Try load (5)	$59 < 70$	$\therefore$ not a maximum.

$\therefore$ Load (4) at L_1 gives maximum.

Bar U_1L_2: Maximum stress occurs with the loading, giving maximum counter-clockwise moment about 0 (intersection of U_1U_2 prolonged and bottom chord). This position may be determined by method of moving up the loads. The distance from L_0 to 0 may be found as follows:

Chord bar U_1U_2 drops 5 ft. in one panel, or 30 ft. in six panels, hence its intersection with the bottom chord is six panel lengths from L_1, or five panel lengths from L_0.

Start with load (2) at L_2 and move up (3).

Increase in moment about 0 of left reaction

$$=(284+2\times 49)\frac{5}{8p}\times 5p+\delta = 382\left(\frac{25}{8}\right)+\delta.$$

Increase in moment about 0 of reaction at floor beam L_1

$$=30\times\frac{5}{p}\times 6p=900\frac{p}{p}=900.$$

Since $382\left(\frac{25}{8}\right)+\delta>900$, load (3) gives greater moment than load (2).

Start with load (3) at L_2 and move up load (4).

$$(284+2\times 54)\frac{5}{8p}\times 5p+\delta<50\times\frac{5}{p}\times 6p.$$

$\therefore$ Load (4) should not be moved up.

Load (3) at L_2 gives maximum.

Bars L_2L_3 and U_1U_2: Maximum stress occurs with loading, giving maximum moment at L_2. This position is found in the usual manner as follows:

Trial Load.	Av'g. Load on Left.	Greater or Less Than	Average Load on Right.	Maximum?
Load (6) to left of L_2......	$\frac{103}{2}$	$<$	$\frac{284+73\times 2-103}{6}=\frac{327}{6}$	No
Load (7) to left of L_2......	$\frac{116}{2}$	$>$	$\frac{430+10-116}{6}=\frac{324}{6}$	Yes
Load (8) to right of L_2.....	$\frac{116}{2}$	$>$	$\frac{440+12-116}{6}=\frac{336}{6}$	No
Load (11) to right of L_2....	$\frac{102}{2}$	$<$	$\frac{284+2\times 105-152}{6}=\frac{342}{6}$	Yes
Load (11) to left of L_2.....	$\frac{122}{2}$	$>$	$\frac{342-20}{6}=\frac{322}{6}$	
Load (12) to right of L_2....	$\frac{102}{2}$	$<$	$\frac{322+2\times 5}{6}=\frac{332}{6}$	Yes
Load (12) to left of L_2.....	$\frac{122}{2}$	$>$	$\frac{332-20}{6}=\frac{312}{6}$	
Load (13) to right of L_2....	$\frac{102}{2}$	$<$	$\frac{312+10}{6}=\frac{322}{6}$	Yes
Load (13) to left of L_2.....	$\frac{122}{2}$	$>$	$\frac{322-20}{6}=\frac{302}{6}$	
Load (14) to right of L_2....	$\frac{122}{2}$	$>$	$\frac{302+10}{6}=\frac{312}{6}$	No

Maximum moment may occur with either load (7), (11), (12) or (13) at L_2. Computations show that load (7) at L_2 gives maximum, as would be expected from inspection of the loading.

Bars U_2L_2 and U_2L_3: Maximum stress occurs with maximum counter-clockwise moment about origin 0 of all forces to left of a vertical section in panel 2–3 (or a diagonal section cutting bars U_1U_2, U_2L_2 and L_2L_3). Determine the position for maximum moment by method of moving up the loads.

Start with load (2) at L_3 and move up load (3).

$$(284+24\times 2)\times\frac{5}{8p}\times 5p+\delta>30\times\frac{5}{p}\times 7p.$$

$$\delta=\frac{2\times 5\times 2.5}{8p}\times 5p.$$

In this case the slight increase in moment due to the term δ is sufficient to cause a larger moment with load (3) than with load (2).

∴ Maximum stress occurs with load (3) at L_3.

Bars L_3L_4 and U_2U_3: Maximum stress occurs with loading causing maximum moment at L_3. This position is found in the usual manner, as follows:

Trial Load.	Av'g Load on Left.	Greater or Less Than.	Average Load on Right.	Maximum?
Load (11) to right of L_3...	$\frac{152}{3}$	<	$\frac{284+80\times 2-152}{5}=\frac{292}{5}$	Yes
Load (11) to left of L_3....	$\frac{172}{3}$	>	$\frac{272}{5}$	
Load (12) to right of L_3...	$\frac{172}{3}$	>	$\frac{272+10}{5}=\frac{282}{5}$	No
Load (13) to right of L_3...	$\frac{192}{3}$	>	$\frac{282-20+10}{5}=\frac{272}{5}$	No

∴ Load (11) at L_3 gives maximum.

Bar U_3L_3: Maximum stress occurs with loading giving maximum moment about O, of forces to left of a diagonal section cutting bars U_2U_3, U_3L_3, and L_3L_4. This position may be determined by method of moving up the loads.

Start with load (1) at L_4 and move up load (2).

$$271\times\frac{8}{8p}\times5p+\delta>10\times8\times\frac{8p}{p}.$$

$\therefore$ Move up load (3).

$$284\times\frac{5}{8p}\times5p+\delta<30\times5\times\frac{8p}{p}.$$

$\therefore$ Load (2) at L_4 gives maximum stress. This loading is evidently consistent with main diagonals, U_2L_3 and U_3L_4, being in action.

Bar U_3L_4: Maximum stress occurs for loading giving maximum positive shear in panel 3–4, and is determined as follows:

Start with load (2) at L_4 and move up load (3).

$$284\times\frac{5}{8p}+\delta>30\times\frac{5}{p}$$

Move up load (4).

$$292\times\frac{5}{8p}+\delta<50\times\frac{5}{p}.$$

$\therefore$ Load (3) at L_4 gives maximum.

Bars $U_4L_5=U_4L_3$, and U_4L_4: Maximum stress occurs for loading giving maximum positive shear in panel 4–5, and is determined as follows:

Start with load (2) at L_5 and move up load (3).

$$232\times\frac{5}{8p}+\delta<30\times\frac{5}{p}.$$

$\therefore$ Load (2) at L_5 gives maximum.

Bar U_5L_6 (counter) $= U_3L_2$. Maximum stress occurs with loading giving maximum clockwise moment about O', the point of intersection of U_5U_6 prolonged and the bottom chord prolonged, of forces to left of vertical section through U_5L_6.

Determine position by method of moving up the loads. Start with load (1) at L_6 and move up load (2).

$$\left(142\times\frac{8}{8p}\right)13p+\delta>10\times\frac{8}{p}\times 8p.$$

Move up load (3).

$$\left(152\times\frac{5}{8p}\right)13p+\delta>30\times\frac{5}{p}\times 8p.$$

Move up load (4).

$$\left(152\times\frac{5}{8p}\right)13p+\delta<50\times\frac{5}{p}\times 8p.$$

∴ Load (3) at L_6 gives maximum moment.

Note that one locomotive followed by uniform load will cause a larger stress than two locomotives.

Bar U_3U_4: Maximum stress occurs for loading giving maximum moment at L_4. This position is found in the usual manner as follows:

Trial Load.	Av'g Load on Left.	Greater or Less Than.	Average Load on Right.	Maximum.
Load (11) to left of L_4........	$\frac{172}{4}$	<	$\frac{222}{4}$	No
Load (12) to left of L_4........	$\frac{192}{4}$	<	$\frac{212}{4}$	No
Load (13) to left of L_4........	$\frac{212}{4}$	>	$\frac{202}{4}$	Yes
Load (14) to right of L_4......	$\frac{212}{4}$	=	$\frac{212}{4}$	Yes

Maximum moment occurs with either load (13) or load (14) at L_4. Computations show that load (13) causes the maximum.

The necessary computations for maximum stresses in all bars are shown in the two tables which follow.

MAXIMUM LIVE WEB STRESSES, IN UNITS OF 1000 LBS.

Cooper's E_{40} Loading

Bar.	Position of Loads.	All Necessary Stress Computations. (L_R = left reaction. V.C. = vertical component.)		
L_0U_1	4 at L_1	Shear $= (16{,}364+284\times 84+84\times 84)\div 200-\frac{480}{25}$ V.C. Stress $=217.2$ Stress $=217.2\times 1.3=$	=	217.2 -282.4
U_1L_1	4 at L_1	Stress $=10\left(\frac{7}{25}\right)+20\frac{(15+20+25+20)}{25}+13\frac{(11+6)}{25}$	=	$+75.6$
U_1L_2	3 at L_2	$L_R=[16{,}364+(284+54)54]\div 200$ Floor beam load at $L_1=230\div 25$ Stress $=[(173.1\times 5p-9.2\times 6p)\div 7p]\times 1.3$	= = =	173.1 9.2 $+150.4$
U_2L_2	3 at L_3	$L_R=[16{,}364+(284+29)29]\div 200$ Floor beam load at L_2 Stress $=(127.2\times 5p-9.2\times 7p)\div 7p$	= = =	127.2 9.2 -81.7
U_2L_3	3 at L_3	V.C. $=(127.2\times 5p-9.2\times 7p)\div 8p$ Stress $=(71.5\times 43)\div 35$	= =	71.5 $+87.8$
U_3L_3	2 at L_4	$L_R=(16{,}364-284)\div 200$ Floor beam load at $L_3=(10\times 8)\div 25$ Stress $=(80.4\times 5p-3.2\times 8p)\div 8p$	= = =	80.4 3.2 -47.1
U_3L_4	3 at L_4	Shear $=[16{,}364+(284+4)4]\div 200-(230\div 25)$ Stress $=(78.4\times 47.2)\div 40$	= =	78.4 $+92.5$
U_4L_3 U_4L_5	} 2 at L_5	Shear $=(8{,}728+232\times 4)\div 200-(10\times 8)\div 25$ Stress $=(45.1\times 47.2)\div 40$	= =	45.1 $+53.2$
U_4L_4	2 at L_5	Stress = V.C. in bar U_4L_5	=	-45.1
U_5L_6 U_5L_7	} 3 at L_6*	$L_R=[3{,}496+142\times 15+20\times 5]\div 200$ Floor beam load at $L_5=230\div 25$ Stress $=[(28.6\times 13p-9.2\times 8p)\div 7p]\times 1.18$	= = =	28.6 9.2 $+50.3$

* One locomotive followed by uniform load.

MAXIMUM LIVE CHORD STRESSES, IN UNITS OF 1000 LBS.

Cooper's E_{40} Loading

Bar.	Position of Loads.	All Necessary Stress Computations. (See previous table for some of the values used in table.)		
L_0L_1 L_1L_2	4 at L_1	$=(217.2\times25)\div30$	=	$+181.0$
L_2L_3	7 at L_2	Mom. at $L_2=[16{,}364+(284+78)78]\times\frac{2}{8}-2155$	=	8995
		Stress $=8995\div35$	=	$+257.0$
U_1U_2	7 at L_2	$257\times25.5\div25$	=	-262.1
L_3L_4	11 at L_3	Mom. at $L_3=[16{,}364+(284+80)80]\times\frac{3}{8}-5848$	=	11208
		Stress $=11208\div40$	=	$+280.2$
* U_2U_3	11 at L_3	Stress $=280.2\times\frac{25.5}{25}$	=	-285.8
U_3U_4	13 at L_4	Mom. at $L_4=[16{,}364+(284+65)65]\times\frac{1}{2}-7668$	=	11856
		Stress $=11856\div40$	=	-296.4

* This stress would be incorrect if the loading used were to throw counter U_3L_2 or L_3U_4 into action. To decide whether this is the case, the shear in panel 2–3 due to this loading may be computed, and the vertical component in top chord U_2U_3 subtracted from it. If the result is positive, or negative, but less (with due allowance for impact) than the dead shear in panel, the counter will not be in action. The computations follow, making use of previous computations and the moment diagram.

$$\text{Shear}=\frac{45{,}484}{200}-116-\frac{8}{25}10-\frac{16+21}{25}13=+89.0.$$

$$\text{V.C.}\quad U_2U_3=280.2\times\frac{10}{50}\qquad=-56.0.$$

$\therefore$ Counter is not in action and stresses are correctly determined.

107. Computation of Stresses. Bridge Trusses with Parabolic Chord. *Uniform Load with Locomotive Excess.* The methods used for the truss considered in the two previous articles were perfectly general and may be used for any non-parallel chord truss. If the panel points on either or both chords lie upon a parabola passing through the end panel points, the truss has, however, certain characteristics which may be taken advantage of in making the computations. Such trusses are not commonly used in railroad bridges, but the same special features occur in certain trussed arches, hence it seems desirable to give an example of the computations for such a truss.

Problem. Determine the maximum stresses in all the members of the truss shown in Fig. 149 with the following loads:

Dead weight of bridge,

1000 lbs. per ft. per truss, top chord	=25,000 lbs. per panel.
400 " " " " " bottom "	=10,000 " " "

Uniform live load,

2000 lbs. per ft. per truss, top chord	=50,000 lbs. per panel.
Locomotive excess, " "	=25,000 lbs.

In the computations for this truss the following points should be noted:

1. When the truss is loaded uniformly throughout its length the ordinates representing the bending moments at the panel

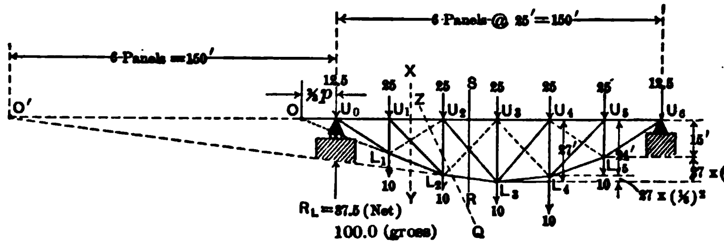


FIG. 149.—Truss with Parabolic Bottom Chord Showing Dead Panel Loads.

points are ordinates of a parabola, hence the bending moment at each panel point divided by the depth of truss at the same panel point is constant.

2. Since the horizontal component of the stress in any bottom chord member equals the moment at a panel point divided by the depth of the truss at the same panel point, the horizontal component of the bottom chord stress under uniform load is constant throughout.

3. For the same reason the top chord stress under uniform load is constant throughout.

4. It follows from 2 and 3 that under uniform load the horizontal components in the diagonals will be zero, and the verticals will all carry compression equal to the top panel load.

5. Since under the dead load (if a uniform load) the stress in the main diagonals is zero, it follows that the live load can always

be so placed as to produce compression in any diagonal, hence if the diagonals are to be tension members counters will be required in every panel.

Dead Stresses. For the given truss the dead stresses in units of 1000 lbs. will be as follows:

$$\begin{array}{lcl}
\text{Top chord, stress} = \frac{1}{8}(1.4)\frac{(150)(150)}{27} & = & -145.8 \\
\text{Bottom chord, horizontal component} & = & +145.8 \\
\text{Diagonals, stress} & = & 0 \\
\text{Verticals, stress} & = & -\ 25.0
\end{array}$$

To confirm the correctness of the conclusions reached for web stresses the diagonal stresses will be computed in the usual manner.

$$\begin{array}{lll}
\text{Shear in panel 1–2} & = 87.5 - 35 & = 52.5 \\
\text{V.C. in bottom chord } L_1L_2 = & \left(\frac{87.5 \times 25}{15}\right)\frac{9}{25} & = 52.5 \\
\text{V.C. in diagonal } U_1L_2 & = 52.5 - 52.5 & = 0. \\
\text{Shear in panel 2–3} & = 87.5 - 70 & = 17.5 \\
\text{V.C. in bottom chord } L_2L_3 = & \left(\frac{87.5 \times 50 - 35 \times 25}{24}\right)\left(\frac{3}{25}\right) & = 17.5 \\
\text{V.C. in diagonal } U_2L_3 & = 17.5 - 17.5 & = 0
\end{array}$$

Counters. Parabolic Trusses. It has been stated that counters are needed in every panel. The truth of this may easily be tested by actual computation. For example, to determine whether counters are required in panel 1–2 assume the section XY, and see if the live load can be so placed as to produce compression in bar U_1L_2. The stress in this bar may be computed by taking moments about the origin O. If a load be placed to the left of XY it will produce a reaction less than itself, and the moment of this reaction about O will be less than the moment of the load itself not only because of its smaller value but because its lever arm is less, hence any load to the left of XY will produce clockwise moment about O of the forces to the left of XY and

thereby cause compression in U_1L_2, therefore a counter will be needed in that panel. As this method is perfectly general, it follows that counters are needed in every panel since the live load can always be placed so as to produce compression in the main diagonals, and the dead stress in these members is zero.

Live Chord Stresses. The maximum live chord stresses occur with the uniform live load extending over the whole truss and can be computed from the dead stresses by multiplying the latter by the ratio of live load to dead load. The chord stresses due to the locomotive excess are as follows:

MAXIMUM STRESSES DUE TO LOCOMOTIVE EXCESS IN UNITS OF 1000 LBS.

Bar.	Position of Load.	Computations.
U_0L_1	E at U_1	H.C. Stress $=\frac{5}{6}25\times\frac{25}{15}=+34.7$
U_0U_1	E at U_1	Stress $=\frac{5}{6}25\times\frac{25}{15}=-34.7$
L_1L_2	E at U_2*	H.C. Stress $=\frac{4}{6}25\times\frac{25}{15}=+27.7$
U_1U_2	E at U_2*	Stress $=\frac{4}{6}25\times\frac{50}{24}=-34.7$
L_2L_3	E at U_3	H.C. Stress $=\frac{1}{2}25\times\frac{50}{24}=+26.0$
U_2U_3	E at U_3	Stress $=\frac{1}{2}25\times\frac{75}{27}=-34.7$

* Note that if E were to be placed at U_1 the counter L_1U_2 would be brought into action hence the horizontal component in bar L_1L_2 would equal $\frac{1}{6}25\times\frac{100}{24}=+17.4$, and the stress in U_1U_2 would equal $\frac{5}{6}25\times\frac{25}{15}= 34.7$.

Live Web Stresses. The maximum live web stresses occur with partial loading. The position of the loads may be determined by the methods previously used. The necessary computations for maximum stresses are given in the following table:

MAXIMUM LIVE WEB STRESSES, IN UNITS OF 1000 LBS

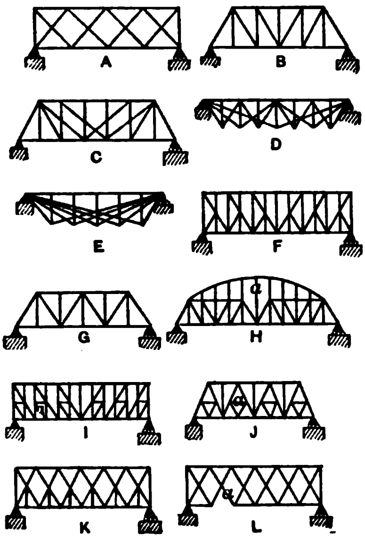
Bar.	Panel Points Loaded with Uniform Load.	Position of E.	Computations, Vertical Components of Maximum Live Web Stresses.
U_1L_1	U_1 or U_1 to U_5 incl.	U_1	$50+25 \qquad = -75.0$
U_1L_2	U_2 to U_5 inclusive	U_2	Shear in panel 1–2 $=\frac{10}{6}50+\frac{4}{6}25 \quad = 100.0$ V.C. in L_1L_2 $=100\times\frac{25}{15}\times\frac{9}{25}= \quad 60.0$ V.C. in U_1L_2 $=100-60 \quad =+40.0$
U_2L_2	U_2 or U_1 to U_5 incl	U_2	$-\ 75.0$
U_2L_3	U_3 to U_5 inclusive	U_3	Shear in panel 2–3 $=\frac{6}{6}50+\frac{3}{6}25 \quad = 62.5$ V.C. in L_2L_3 $=62.5\times\frac{50}{24}\times\frac{3}{25}= \quad 15.6$ V.C. in U_2L_3 $=62.5-15.6 \quad =+46.9$
U_3L_3	U_3 or U_1 to U_5 incl.	U_3	$-\ 75.0$
U_3L_4	U_4 and U_5	U_4	Shear in panel 3–4 $=\frac{3}{6}50+\frac{2}{6}25 \quad = 33.3$ V.C. in L_3L_4 $=33.3\times\frac{75}{27}\times\frac{3}{25}= \quad 11.1$ V.C. in U_3L_4 $=33.3+11.1 \quad =+44.4$
U_4L_5	U_5	U_5	Shear in panel 4-5 $=\frac{1}{6}75 \quad = 12.5$ V.C. in L_4L_5 $=12.5\times\frac{100}{24}\times\frac{9}{25}=18.75$ V.C. in U_4L_5 $=12.5+18.75=+31.25$

For actual locomotive loads the computations for this truss should present no more difficulty than for the truss of the previous example.

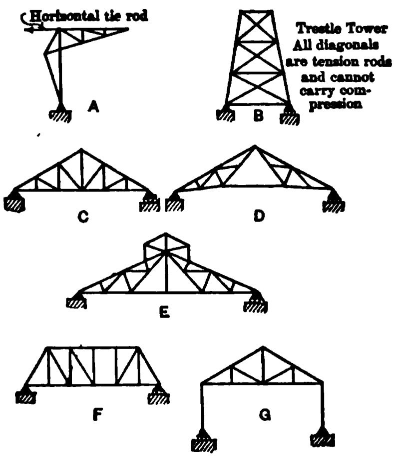
PROBLEMS

41. State which of the trusses shown in the figure are statically determined with respect to the inner forces, and give reasons. Points of intersections of wet members should not be considered as joints.

42. State which of the structures shown in the figure are statically undetermined with respect to the inner forces, and give reasons.



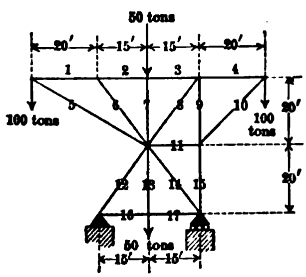
PROB. 41.



PROB. 42.

43. *a.* Compute by the analytical method of joints the vertical components in all diagonal members, and the actual stress in all other members of this structure. Tabulate results in order according to bar numbers. Designate tension by (+) and compression by (−).

b. Determine stress in all members by graphical method of joints (Bow's Notation). Tabulate results in same order as in *a.*



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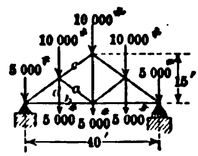
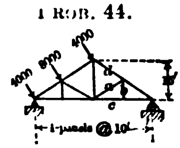


PROB. 44.



PROB. 45.

44. Compute by method of moments the stress in bars *a*, *b*, *c* and *d*, and state whether stress is tension or compression.

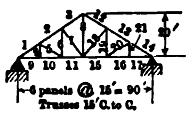
45. *a.* Compute by method of moments the stress in bars *a*, *b*, *c* and *d*; and state whether tension or compression.

b. Same as *a*, but direction of reaction is not fixed by rollers. (Assume both reactions to act parallel to direction of applied loads.)

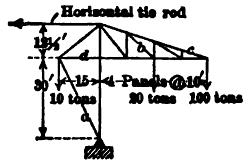
46. Compute maximum stress in each member due to following loads applied at top chord:

1. Dead, 30 lbs. per horizontal square foot.
2. Snow, 20 lbs. per horizontal square foot.
3. Wind, 30 lbs. per sq. ft. normal to surface.

Tabulate stresses for each kind of loading, and determine maximum stresses, arranging results according to bar numbers as given on diagram. Indicate tension thus (+) and compression thus (−).



PROB. 46.



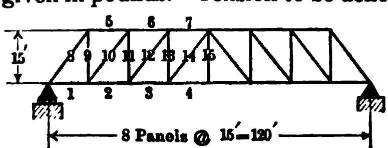
PROB. 47.

47. Compute stresses in tons and state whether tension or compression for the following bars:

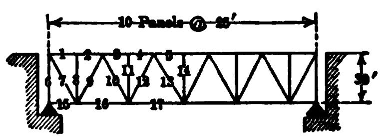
Bar *a*, by method of joints.
Bar *b*, by method of moments.
Bar *c*, by method of joints.
Bar *d*, by method of moments.

48. Uniform live load, 2000 lbs. per foot, on top chord.
Locomotive excess, 20,000 lbs. on top chord.
Impact by formula (7).
Dead load, top chord, 600 lbs. per foot.
Dead load, bottom chord, 200 lbs. per foot.

Determine panels in which counters are needed and compute maximum stress in each member of this truss. Number bars as shown in figure and arrange results in order according to bar numbers. Stresses to be given in pounds. Tension to be denoted by (+) and compression by (−).



PROB. 48.



PROB. 49.

49. Uniform live load, 2000 lbs. per foot on top chord.
Locomotive excess, 20,000 lbs. top chord.
Dead load, 1000 lbs. per foot top chord.
Dead load, 500 lbs. per foot bottom chord.

No counters are to be used. Compute maximum stresses of both kinds in all members of this truss. (Rules as to arrangement of results, etc., as in previous problems.)

CHAPTER VII

BRIDGE TRUSSES WITH SECONDARY WEB SYSTEMS, INCLUDING THE BALTIMORE AND PETTIT TRUSSES

108. Secondary Systems Described. The bridge trusses heretofore treated have all been of such simple types that the application of the ordinary methods of joints, moments, and shear required no special explanation. For spans of considerable length, however, the frequent subdivision of the main panels and the addition of a secondary set of diagonals and verticals produce complications the effect of which will be explained in this chapter. The Baltimore and Pettit trusses, illustrated by Figs. 132 and 134, are the common forms of such trusses and will alone be considered. An examination of one of these trusses shows that the stresses in the secondary verticals may easily be determined by the method of joints, the real complication occurring in the secondary diagonal stresses.

In order to study the stress in one of these diagonals, consider the portion of such a truss shown in Fig. 150, and let the problem be the determination of the stress in diagonal L_2M_3. In order that the case may be perfectly general, let it be assumed that the dead loads are applied at the middle as well as top and bottom panel points, although such an accurate division of the dead panel loads is not generally required.

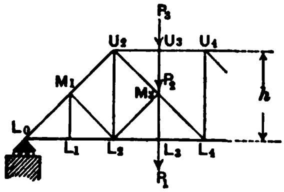


FIG. 150.

An examination of the forces acting at joint M_3 shows that the function of the secondary diagonal M_3L_2 is to support the main diagonal, U_2L_4, under the loads P_1, P_2 and P_3, which, if the secondary diagonal were not inserted, would cause the member U_2L_4 to bend and thereby produce the collapse of the truss. If no loads be applied at the secondary panel points U_3, M_3 or L_3,

[1] Secondary members are bars which are stressed by loads acting at specific panel points. Principal members are stressed by loads acting at any panel point.

there will be no stress in either the secondary diagonal M_3L_2, or the secondary verticals U_3M_3 and M_3L_3, since these members take no part in the transmission to the abutment of the loads at other panel points. It is therefore necessary to consider only the loads P_1, P_2 and P_3, in determining the stresses in the secondary members meeting at the joint M_3, and the effect of these members upon the main diagonal stresses. The stresses in the secondary verticals are evidently equal to the panel loads applied at their ends; that is, the compression in $U_3M_3=P_3$, which ordinarily is merely the dead weight of the top chord acting at this point, and the tension in $M_3L_3=P_1$. This is equivalent, so far as the secondary diagonal is concerned, to the application at M_3 of a resultant downward vertical force equal to $P_1+P_2+P_3$. For simplicity this resultant will hereafter be called R and considered as acting directly at M_3. The stress in M_3L_2 may then be computed by the method of joints by resolving the force R along two axes coincident with U_2L_4 and L_2M_3. With this stress known, the effect of R upon U_2M_3 may also be readily determined by the same method. The stress in M_3L_4 is evidently unaffected by the secondary system and may be determined by the method of shear in the usual manner, since its vertical component equals the shear in panel 3–4.

While the method of joints for this case presents no special difficulty, the method of moments is much simpler, as is seen from the following discussion.

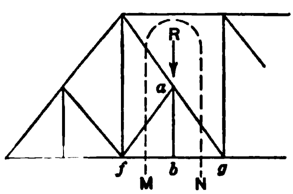


FIG. 151.

Consider the truss shown in Fig. 151 and apply the method of moments to the forces acting on the portion of the truss lying within the curved section MN. All the bars cut by this section except fa meet at joint g which should be taken as the origin of moments. If we now let V = vertical component of the stress in bar fa, assuming it to be tension and write the equation for moments about point g the following expression is obtained:

$$R(bg)+V(fg)=0$$

hence

$$V=-R\frac{bg}{fg}=-(P_1+P_2+P_3)\frac{bg}{fg}$$

It is evident that the same method could be applied if the secondary diagonal were to extend from M_3 to U_4, Fig. 150, instead of from M_3 to L_2, but the section MN should in this case cut the top chord instead of the bottom chord and be inverted. The numerical value of the stress in the bar would be the same as for that just found but it would be in tension instead of compression.

The following proposition may now be stated.

The *vertical component* of the *compression* in bar (1) in the case shown by Fig. 152, or the *vertical component* of the tension in bar (2) in the case shown by Fig. 153 $=\frac{P_1+P_2+P_3}{2}$.

It follows from the above rule that the vertical component of the maximum stress in a secondary diagonal of a Baltimore truss with equal panels and horizontal chords equals one-half the maximum panel load.

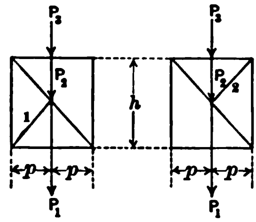


FIG. 152. FIG. 153.

With the vertical component in the secondary diagonal known the vertical component in the main diagonal in the same panel may be found by subtracting this value from the shear in the panel if the bars be as shown in Fig. 152, or by adding it to the shear for the case shown in Fig. 153, provided in both cases that the shear is positive. In case the shear is negative, the question of counters must be investigated in accordance with the methods of the following article.

The demonstration just given is simple, but the results apply to trusses with parallel chords only. The same method may, however, be readily applied to trusses with non-parallel chords. The demonstration which follows is given, however, in order to illustrate a method which is often very useful in determining bar stresses in certain forms of trusses.

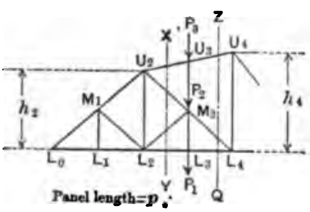


FIG. 154.

This method consists in first deriving an expression for the sum of the horizontal components in L_2M_3 and L_2L_3, Fig. 154, called hereafter for convenience H.C. $(L_2M_3+L_2L_3)$, and then sub-

tracting from it the horizontal component in L_2L_3 called H.C. (L_2L_3).

Let M_2 = the moment of the forces to the left of XY about U_2.

Let M'_2 = the moment of the forces to the left of ZQ about U_2.

Then $$\text{H.C. } (L_2M_3+L_2L_3)=\frac{M_2}{h_2}$$

and $$\text{H.C. } (L_3L_4)=\text{H.C. } (L_2L_3)=\frac{M'_2}{h_2}.$$

$$\therefore \quad \text{H.C. } (L_2M_3+L_2L_3)-\text{H.C. } (L_2L_3)=\text{H.C. } (L_2M_3)=\frac{M_2-M'_2}{h_2}.$$

The only difference between M_2 and M'_2 is the moment of the forces P_1, P_2 and P_3, acting between XY and ZQ, since otherwise the forces to the left of the two sections are identical, hence

$$\frac{M_2-M'_2}{h_2}=(P_1+P_2+P_3)\frac{p}{h_2}=\text{H.C. } (L_2M_3).$$

For the case shown by Fig. 154, M'_2 will be larger than M_2, hence the above result will be negative, showing compression in the secondary diagonal L_2M_3.

If the secondary diagonal be a tension member, as shown in Fig. 155, instead of the compression bar of Fig. 154, the same general method applies; but M_2 and M'_2 should for this case be replaced by M'_4 and M_4, the moments about L_4 of the forces to the left of ZQ and XY respectively, and the expression should have h_4 in the denominator instead of h_2. The following expression results:

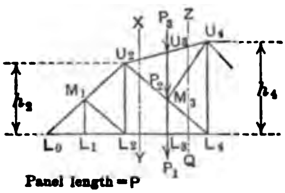


FIG. 155.

$$\text{H.C. } (U_3U_4+M_3U_4)-\text{H.C. } (U_3U_4)=\frac{M'_4-M_4}{h_4}.$$

$$\therefore \quad \text{H.C. } (M_3U_4)=(P_1+P_2+P_3)\frac{p}{h_4}.$$

This result will be positive, thereby indicating tension in the bar.

It should be noticed that in all these cases the intermediate panel point has been so located as to divide the main diagonal at the *centre*, and the two halves of the latter member have been in

the same straight line. Moreover, the chords have been straight between the panel points.

Were these conditions not to exist the demonstration would not be true. For example, if the members were to be as shown in Fig. 156, it would be necessary to determine the value of the stress in the secondary diagonal by a special method. A general equation for this case will not be given, but for any given truss the stress in U_2M_3 may be readily obtained by the method of moments, using for origins L_2 and m, with sections XY and ZQ as before. It should be noticed that in such a case the stress in U_2M_3 is not only a function of the panel loads P_1, P_2 and P_3, but also of the loads at all other panel points, since the moment about m of the outer forces to the left of XY differs from the moment of these same forces about L_2.

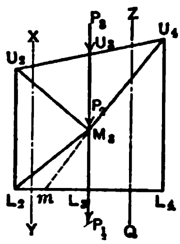


FIG. 156.

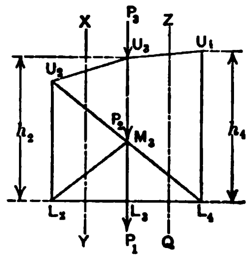


FIG. 157.

A somewhat similar case is shown by Fig. 157, where the top chord is not straight between main panel points. In this case the stress in the secondary diagonal is a function of the stress in the vertical M_3U_3 as well as of the panel loads P_1, P_2 and P_3. Since the stress in M_3U_3 is a function of the upper chord stresses it is in consequence affected by all the loads on the structure. The simplest method of solution for this case for any given loading is to combine the stress in U_3M_3 with the panel load P_3, and then proceed as if the top chord bar were straight between U_2 and U_4. The stress in U_3M_3 may readily be obtained by applying the method of joints to the forces acting at U_3, having first found the horizontal components of the top chord stresses in U_3U_2 and U_3U_4 in the usual manner by the method of moments, and from these their vertical components.

The stress in a main diagonal, such as U_2M_3 of a truss, like that shown in Fig. 154, can be easily computed, provided the

stress in the secondary diagonal is known. It should be observed, however, that the stress in the main diagonal depends not only upon the shear and the stress in the secondary member, but also upon the vertical component in the top chord. This case is more complicated than for the parallel chord truss, but is fully illustrated by the example given in the following article.

109. Computation of Maximum Stresses in Pettit Truss. *Dead Loads and Concentrated Load System.*

Problem. Let the problem be the computation of the maximum stresses in all bars of the truss shown in Fig. 158 for the following loads.

Dead load on top chord per horizontal foot = 2250 lbs. per truss = 68,000, lbs. per panel (approx.).

Dead load on bottom chord = 3500 lbs. per foot per truss = 105,000 lbs. per panel.

Live load. Cooper's E_{50} standard loading.

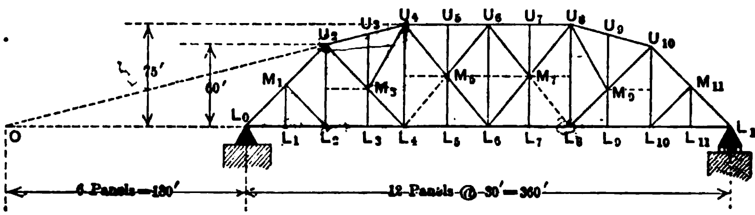


FIG. 158.—Double Track Railroad Bridge. All Diagonals Tension Members.

In this truss the dotted horizontal members are used to support the main verticals against buckling, and are subjected to secondary stresses only; a common device in long-span trusses. The dotted diagonals represent counters, and are not in action under dead loads. In the computations which follow, the moment diagram for Cooper's E_{40} loading given in Art. 52 has been used, and the stresses for E_{50} obtained by multiplying by the ratio $\frac{50}{40}$. All units are in thousands of pounds.

Index Stresses. In determining the index stresses, it is necessary, as in the previous example, to first determine the vertical component in the inclined top chord bar, and to correct the diagonal stresses to conform to the slope of the end diagonals.

As the stresses in the secondary members are independent of the stresses in the main members, it is advisable to write these first. For the other members the usual process will be pursued of beginning at the centre and working towards the end, checking with the reaction at the end and with the chord stress as computed by moments at the centre.

The index stresses are given in Fig. 159, and the necessary

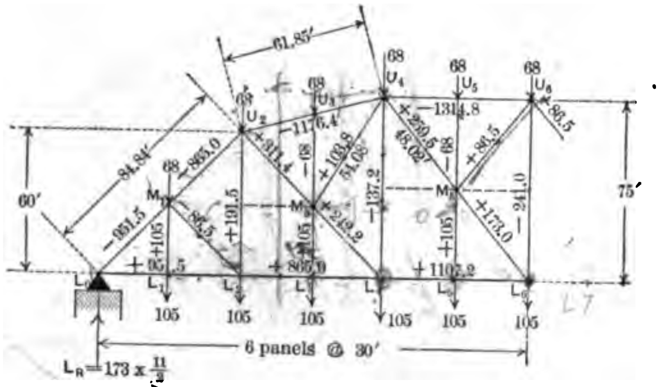


FIG. 159.—Index Stresses and Dead Panel Loads for Truss Shown in Fig. 158.

computations for bars in which the index stresses are at all complicated follow. In determining the index stresses the diagonals sloping at 45° are taken as standard.

V.C. in $M_3U_4 = 173 \times \frac{30}{75} \times \frac{45}{30}$ $= 103.8$

V.C. in $U_2U_4 = \left(\frac{951.5 \times 120 - 173 \times 150}{75}\right)\frac{15}{60}$ $= 294.1$

V.C. in $U_4L_4 = 259.5 + 103.8 + 68.0 - 294.1$ (method of joints) $= 137.2$

V.C. in $U_2M_1 = 191.5 + 311.4 + 294.1 + 68.0$ (method of joints) $= 865.0$

Corrected index stress in M_3U_4, for use in determining index stress in bar U_4U_5, $= 103.8 \times 30/45 = 69.2$

Corrected index stress in U_4M_5, for use in determining index stress in bar U_4U_5, $= 259.5 \times 60/75 = 207.6$

To check top chord stresses determine combined horizontal component in U_5U_6 and M_5U_6 by dividing centre moment by

centre height, and add to the value thus obtained the horizontal component in M_5U_6.

$$\text{Stress in } U_4U_5 = \frac{951.5\times180-173\times5\times90}{75}+86.5\times\frac{30}{37.5}=1314.8.$$

This equals the index stress in U_4U_5, as should be the case, since the latter was determined for diagonals sloping at 45°.

Dead Stresses. The actual dead stresses are given in the following table in which the columns headed "ratio" give the length of each web member divided by its vertical projection and of each chord member divided by its horizontal projection.

DEAD STRESSES IN UNITS OF 1000 POUNDS

Bars.	Index Stress.	Ratio.	Stress.	Bars.	Index Stress.	Ratio.	Stress.
L_0M_1	− 951.5	1.414	−1345.4	L_1M_1	+105.0	1.000	+105.0
M_1U_2	− 865.0	1.414	−1223.1	L_3M_3	+105.0	1.000	+105.0
U_2U_3	−1176.4	1.031	−1212.9	L_5M_5	+105.0	1.000	+105.0
U_3U_4	−1176.4	1.031	−1212.9	U_3M_3	− 68.0	1.000	− 68.0
U_4U_5	−1314.8	1.000	−1314.8	U_5M_5	− 68.0	1.000	− 68.0
U_5U_6	−1314.8	1.000	−1314.8	M_1L_2	− 86.5	1.414	−122.3
L_0L_1	+ 951.5	1.000	+ 951.5	L_2U_2	+191.5	1.000	+191.5
L_1L_2	+ 951.5	1.000	+ 951.5	U_2M_3	+311.4	1.414	+440.3
L_2L_3	+ 865.0	1.000	+ 865.0	M_3L_4	+242.2	1.414	+342.5
L_3L_4	+ 865.0	1.000	+ 865.0	M_3U_4	+103.8	1.202	+124.8
L_4L_5	+1107.2	1.000	+1107.2	U_4L_4	−137.2	1.000	−137.2
L_5L_6	+1107.2	1.000	+1107.2	U_4M_5	+259.5	1.280	+332.2
				M_5U_6	+ 86.5	1.280	+110.7
				M_5L_6	+173.0	1.280	+221.4
				L_6U_6	−241.0	1.000	−241.0

Counters. Before computing the live stresses, and even before determining the position of live loads for maximum stresses, it is necessary to decide in what panels counters are required.

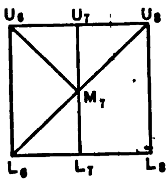


FIG. 160.

Panels 4–5 and 7–8. Evidently counters will be needed in these panels if the resultant shear due to live, dead and impact in panel 7–8 is ever positive. The application of the method of moving up the loads shows that load (2) at L_8 gives maximum positive shear in this panel. Its magnitude per truss for E_{50} equals

$$\frac{5}{4}\times2\left[\frac{16364+(284+19)19}{360}\right]-\frac{5}{4}\times2\times\frac{80}{30}=146.9.$$

If impact be computed by formula (7), its value will be $\frac{300}{428}146.9 = 102.9$, hence the live shear plus impact $= 102.9 + 146.9 = 249.8$. The dead shear in this panel equals -259.5, but the difference between this and the live shear plus impact is so small that the counter should be used.

Panels 2–3 and 9–10. Counters will be needed in these panels if the live compression plus impact in bar M_9U_{10} exceeds its dead tension. The position of loads which will give the maximum compression in M_9U_{10} will be that which will give maximum clockwise moment of forces to left of vertical section through panel 9–10 about O', the intersection of top chord bar $U_8U_9U_{10}$ prolonged and the bottom chord prolonged.

To determine this position start with load (1) at L_{10} and move up load (2), using p to represent the panel length,

$$152 \times \frac{8}{12p} \cdot 18p + \delta > 10 \times \frac{8}{p} \times 9p. \quad \therefore \quad \text{move up load (2).}$$

Now try moving up load (3).

$$172 \times \frac{5}{12p} \cdot 18p + \delta > 30 \times \frac{5}{p} \cdot 9p.$$

$\therefore$ load (3) at L_{10} gives maximum.

V.C. live stress in bar M_9U_{10} with load (3) at L_{10} for E_{50}

$$= 2 \times \frac{5}{4} \times \frac{1}{300}\left[(7668 - 192) \times \frac{18p}{12p} - \frac{230}{p} \times 9p\right] = 76.2.$$

This value is so much less than the vertical component of the dead stress in the bar that no counter is needed.

Position of Loads for Maximum Live Stress in all Members:

BAR U_2M_3. Load for maximum moment about O of loads to left of a vertical section through panel 2–3.

Start with load (2) at L_3 and move up load (3).

$$(284 + 169 \times 2)\frac{5}{12p} \times 6p + \delta > 30 \times \frac{5}{p} \times 8p.$$

Move up load (4).

$$(284+174\times 2)\frac{5}{12p}\times 6p+\delta<50\times\frac{5}{p}\times 8p.$$

$\therefore$ Load (3) at L_3 gives maximum.

BAR M_3L_4. Let M_4/h_4 = moment about U_4 of forces to left of a vertical section through panel 3–4 divided by height of truss at L_4. Let M_2/h_2 = moment about U_2 of forces to left of a vertical section through panel 2–3 divided by height of truss at L_2. Since the horizontal component of the stress in $M_3L_4 = M_4/h_4 - M_2/h_2$, the position of loads for maximum stress in the bar is that giving the maximum values of this quantity. Fig. 161

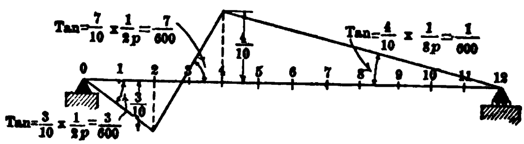


FIG. 161.—Influence Line for Horizontal Component in M_3L_4.

shows the influence line for the horizontal component in this bar, and shows that one of the loads should lie at L_4. To determine the position for maximum stress use the method of moving up the loads, multiplying the loads to right of L_4 by the product of the distance moved and the tangent 1/600, and those in panels 2–3 and 3–4 by the product of the distance moved and the tangent 7/600.

Start with load (3) at L_4 and move up load (4).

$$(234+144\times 2)\frac{5}{600}+\delta>50\times 5\times\frac{7}{600}.$$

Move up load (5).

$$(214+149\times 2)\frac{5}{600}+\delta>70\times 5\times\frac{7}{600}.$$

Move up load (6).

$$(194+154\times 2)\frac{9}{600}+\delta<90\times 9\times\frac{7}{600}.$$

Load (5) at L_4 gives a maximum.

It is possible that this bar may be brought into compression by loads coming on from left, hence the position giving maximum compression should be determined.

Start with load (2) at L_2 and move up load (3), bringing loads on from left,

$$142 \times 5 \times \frac{3}{600} + \delta > 30 \times 5 \times \frac{7}{600}.$$

Move up load (4).

$$142 \times 5 \times \frac{3}{600} + \delta > 50 \times 5 \times \frac{7}{600}.$$

Move up load (5).

$$142 \times 5 \times \frac{3}{600} + \delta < 70 \times 5 \times \frac{7}{600}.$$

$\therefore$ Load (4) at L_2 gives a maximum.

BAR U_4M_5. Load for maximum shear in panel 4–5.

Start with load (2) at L_5 and move up load (3):

$$(284 + 109 \times 2)\frac{5}{12p} + \delta > 30 \times \frac{5}{p}.$$

Move up load (4).

$$(284 + 114 \times 2)\frac{5}{12p} + \delta < 50 \times \frac{5}{p}.$$

$\therefore$ Load (3) at L_5 gives maximum.

BAR M_5L_6. Load to give the maximum value of the resultant of the positive shear in panel 5–6 and the vertical component in bar M_5U_6.

Start with load (3) at L_6 and move up load (4).

$$(284 + 84 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}\left(50 \times \frac{5}{p}\right).$$

Move up load (5).

$$(284 + 89 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}\left(70 \times \frac{5}{p}\right).$$

Move up load (6).

$$(284 + 94 \times 2)\frac{9}{12p} + \delta < \frac{1}{2}\left(90 \times \frac{9}{p}\right).[1]$$

Load (5) at L_6 gives maximum.

[1] The right-hand side of this inequality equals the increment in the sum of the panel load at L_4 and the vertical component of the stress in the secondary

BAR M_7L_8. If this bar be in action the condition shown in Fig. 163 will exist. Place loads so that the sum of the positive

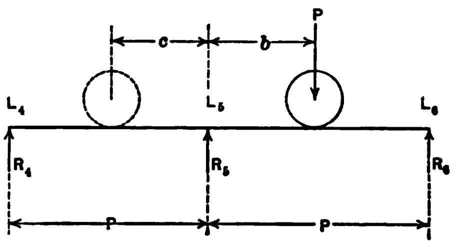


FIG. 162.

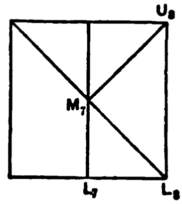


FIG. 163.

shear in panel 7–8 and the vertical component in bar M_7U_8 will be a maximum.

Start with load (2) at L_8 and move load (3).

$$(284+19\times 2)\frac{5}{12p}+\delta>\frac{1}{2}\ 30\times\frac{5}{p}.$$

Move up load (4).

$$(284+24\times 2)\frac{5}{12p}+\delta>\frac{1}{2}\ 50\times\frac{5}{p}.$$

Move up load (5).

$$(284+29\times 2)\frac{5}{12p}+\delta<\frac{1}{2}\ 70\times\frac{5}{p}.$$

Load (4) at L_8 gives a maximum.

---

diagonal M_8U_8 due to the movement of the loads. If no load passes L_6 it is obvious that this change equals one-half the sum of the product of the loads moving in panel 5–6 and the distance which they move. That this is also true, provided no load passes L_4, may be readily proven as follows:

Let the original position of a load P be as shown by the full circle in Fig. 162, and assume that in moving the loads, P passes L_5 to the position shown by the dotted circle.

The following equations may then be written:

Original position of loads: $$R_4+\frac{R_5}{2}=\frac{P(p-b)}{2p}.$$

Second position of loads: $$R_4+\frac{R_5}{2}=P\frac{c}{p}+\frac{P}{2p}(p-c).$$

The increase in $R_4+\frac{R_5}{2}=P\frac{c}{p}+P\frac{(p-c)}{2p}-\frac{P(p-b)}{2p}=\frac{P(b+c)}{2p}$.

This demonstration applies equally well to two corresponding panels in any other position of the truss.

Bars L_0M_1, L_0L_1, L_1L_2. Load for maximum shear in panel 0–1. Start with load (3) at L_1 and move up load (4).

$$(284+234\times 2)\frac{5}{12p}+\delta>50\times\frac{5}{p}.$$

Move up load (5).

$$(284+239\times 2)\frac{5}{12p}+\delta<70\times\frac{5}{p}.$$

∴ Load (4) at L_1 gives maximum.

BARS M_1U_2, L_2L_3 and L_3L_4. Load for maximum moment at U_2.

Try load (7) at L_2, $624/10>116/2$. Not a maximum.

Try load (8) at L_2, $636/10>116/2$ and $623/10<129/2$. A maximum.

Try load (9) at L_2, $633/10<129/2$. Not a maximum.

∴ Load (8) at L_2 gives a maximum.

BAR U_2L_2. This bar is really a part of the secondary system and is affected by loads at L_1 and L_2 only. The influence line for this bar is shown by Fig. 164, and has the same form as the influence line for moment at a point 30 feet from the right end of an end-supported 90-ft. span; hence the criterion for maximum moment may be applied to determine the position of loads which should be brought on from the left.

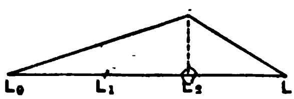


FIG. 164.—Influence Line for Stress in U_2L_2.

Try load (3) at L_2, $142/60>50/30$. Not a maximum.

Try load (4) at L_2, $142/60>70/30$. Not a maximum.

Try load (5) at L_2, $142/60<90/30$. A maximum.

∴ Load (5) at L_2 gives a maximum.

BARS U_2U_3 and U_3U_4. Load for maximum moment about L_4 of forces to left of vertical section through panel 2–3. The influence line for the horizontal component of the stress in these bars is shown in Fig. 165. Evidently for a maximum one of the loads should lie at L_3.

While the influence line for the stress in this case is not composed of two straight lines and the criterion for maximum moment cannot be applied, it is evident that the loads will lie somewhat

as for the ordinary case of maximum moment at a panel point, and one of the second-engine loads will probably give the maximum.

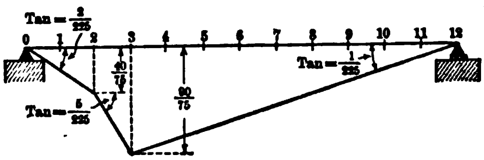


FIG. 165.—Influence Line for Horizontal Component in U_2U_3 and U_3U_4.

The following expression for the change in the stress may be written, using the method of moving up the loads.

Start with load (11) at L_3 and move up load (12):

$$(112+225\times 2)\frac{5}{225}+\delta>56\times 5\times\frac{5}{225}$$
$$+13\left(3\times\frac{5}{225}+2\times\frac{2}{225}\right)+103\times 5\times\frac{2}{225}.$$

Move up load (13).

$$(92+230\times 2)\frac{5}{225}+\delta<63\times 5\times\frac{5}{225}$$
$$+13\left(4\times\frac{5}{225}+1\times\frac{2}{225}\right)+116\times 5\times\frac{2}{225}.$$

$\therefore$ Load (12) at L_3 gives maximum.

BAR U_4L_4. *Assuming that counter M_5L_4 is not in action,* the influence line for stress in this bar will be as given in Fig. 166,

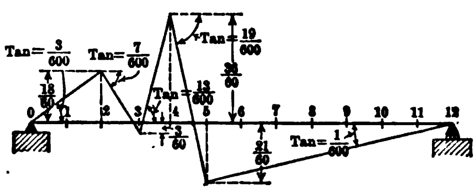


FIG. 166.—Influence Line for Stress in U_4L_4.

and shows that for maximum compression the load should come on from the right, and for maximum tension from the left.

Position for maximum compression, load coming on from right:

Start with load (1) at L_5 and move up load (2), making use of the tangents to the influence line, as was done with bar M_3L_4.

$$(274+101\times 2)\frac{8}{600}+\delta>10\times 8\times\frac{19}{600}.$$

Move up load (3).

$$(254+109\times 2)\frac{5}{600}+\delta<30\times 5\times\frac{19}{600}.$$

∴ Load (2) at L_5 gives maximum compression.

Position for maximum tension, loads coming on from left:

For this case heavy loads should be placed at both L_4 and L_2. These panel points are 60 ft. apart, hence if the heavy loads of the first locomotive are placed near L_4, the heavy loads of the second locomotive will be located near L_2, this giving a favorable position for maximum stress.

Start with load (2) at L_4 and move up load (3):

$$86\times 5\times\frac{13}{600}+(92+2\times 19)\times 5\times\frac{3}{600}+20\times 1\times\frac{3}{600}$$
$$+\delta>30\times 5\times\frac{19}{600}+56\times 5\times\frac{7}{600}+20\times 4\times\frac{7}{600}.$$

Move up load (4).

$$79\times 5\times\frac{13}{600}+(72+2\times 24)\times 5\times\frac{3}{600}+20\times 1\times\frac{3}{600}$$
$$+\delta<50\times 5\times\frac{19}{600}+63\times 5\times\frac{7}{600}+20\times 4\times\frac{7}{600}.$$

∴ Load (3) at L_4 gives maximum.

The above condition for maximum stress will not be correct if in either case counter M_5L_4 be in action. That this bar is not in action for the position of loads giving maximum compression is, however, evident from inspection.

For the position for maximum tension the negative shear in panel 4–5 is given by the following expression:

Shear in panel 4–5, load (3) at L_4, loads coming on from left

$$=2\times\frac{5}{4}\left[\frac{16364+(284+24)24}{360}-\frac{230}{30}\right]=145.7.$$

This is considerably smaller, even after impact is added, than the positive dead shear in the panel and the counter will not be in

action; hence the assumed condition is consistent with the position of the loads as determined.

Bars U_4U_5 and U_5U_6. Load for maximum moment about L_6 of loads to left of vertical section through panel 4–5. The influence line for the stress in this case consists of three straight

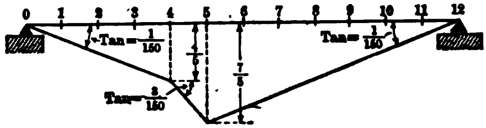


FIG. 167.—Influence Line for Stress in U_4U_5 and U_5U_6.

lines, as shown in Fig. 167, and shows that the maximum stress occurs with one of the loads at L_5.

Start with load (14) at L_5 and move up load (15), making use of the tangents to the influence line.

$$(52+180\times 2)\frac{9}{150}+\delta>142\times\frac{9}{150}+10\left(\frac{2}{150}+7\times\frac{3}{150}\right)$$
$$+80\times 9\times\frac{3}{150}.$$

Move up load (16).

$$(39+189\times 2)\frac{5}{150}+\delta<152\times\frac{5}{150}+93\times 5\times\frac{3}{150}.$$

$\therefore$ Load (15) at L_5 gives maximum.

Bars L_4L_5 and L_5L_6. Load for maximum moment at U_4 assuming counter L_4M_5 to be out of action.

Try load (14) to left of L_4, $\frac{232}{4}<\frac{472}{8}$. Not a maximum.

Try load (15) to left of L_4, $\frac{245}{4}>\frac{477}{8}$. A maximum.

Try load (16) to right of L_4, $\frac{245}{4}>\frac{487}{8}$. Not a maximum.

$\therefore$ Load (15) at L_4 gives a maximum.

The shear in panel L_4L_5 for this condition

$$=2\times\frac{5}{4}\left[\frac{16364+(284+219)219}{360}-245-\frac{58}{30}13-\frac{18}{30}4\tfrac{1}{2}\right]=+196.5.$$

Hence counter L_4M_5 is not in action for this loading.

Bar U_6L_6. Two cases must be considered for this bar. These are shown in Fig. 168.

Case I. Maximum stress, if this case exists, will occur with the loading giving the maximum value of the algebraic sum of the positive shear on section XY and the vertical component in diagonal M_5U_6.

Try load (2) at L_7 and move up load (3).

$$(284+49\times 2)\frac{5}{12p}+\delta>30\times\frac{5}{p}.$$

Move up load (4).

$$(284+54\times 2)\frac{5}{12p}+\delta<50\times\frac{5}{p}.$$

$\therefore$ Load (3) at L_7 gives a maximum.

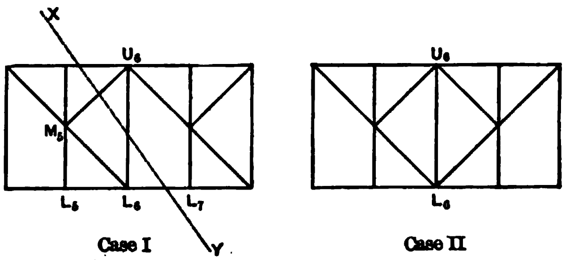


FIG. 168.

Case II. The maximum stress, if this case exists, cannot exceed twice the vertical component of the maximum stress in one of the secondary diagonals; i.e., it will not exceed the maximum panel load. Since the stress in Case I is likely to be greater than this limiting value, the position of loads should not be determined until after the stress for Case I has been computed. If it then becomes necessary to determine the position, the method of influence lines will be used.

Bars M_1L_1, M_1L_2, M_3L_3, U_4M_3, M_5L_5 and U_6M_5. Maximum stress in these bars is a function of the maximum load at a secondary panel point. This has the same value in all cases, and may be found for any one of these panel points, such as L_1, by placing the loads so as to give the maximum moment at the centre of a 60-ft. span.

Try load (12) at L_1, $86>56$ and $66<76$, hence a maximum.

Try load (13) at L_1, $79>63$ and $72<83$, hence a maximum.

Try load (14) at L_1, $72>70$ and $52<90$, hence a maximum.

$\therefore$ Maximum stress occurs with either load (12), (13), or (14) at a secondary panel point. (Note that load (14) gives same moment as load (5).)

MAXIMUM LIVE STRESSES IN MAIN DIAGONALS IN UNITS OF 1000 POUNDS

This table shows all necessary computations. (Note that $16{,}364 \div 360 = 45.45$.)

Bar.	Position of Loads.	Computations.
U_2M_3	3 at L_3 Max. Tension.	Shear in panel 2–3 $=\frac{16{,}364+(284+174)174}{360}-\frac{230}{30}=$ $=266.8-7.7 \qquad = +259.1$ Vert. Comp. in U_2U_3 $=\frac{15}{60}(266.8\times 4-7.7\times 2)\frac{30}{75} \qquad = -105.2$ Vert. Comp. in U_2M_3 in tons for E_{40} $\qquad = +153.9$ Tension for $E_{50}=153.9\times 1.414\times\frac{5}{4}\times 2 \qquad = +544.0$
M_3L_4	5 at L_4 Max. Tension.	Shear in panel 3–4 $=\frac{16{,}364+(284+154)154}{360}-\frac{830}{30}$ $232.8-27.7 \qquad = +205.1$ Vert. Comp. in U_3U_4 $=\frac{15}{60}(232.8\times 4)\frac{30}{75} \qquad = -\ 93.1$ Vert. Comp. in M_3U_4 $=\frac{45}{30}\left(\frac{830}{30}\right)\left(\frac{30}{75}\right) \qquad = +\ 16.6$ Vert. Comp. in M_3L_4 in tons for E_{40} $=205.1-93.1+16.6 \qquad = +128.6$ Tension for $E_{50}=\ 128.6\times 1.414\times\frac{5}{4}\times 2 \qquad = +454.6$
M_3L_4	4 at L_7 Max. Comp., Loads coming on from left.	Shear in panel 3–4 $=\frac{8728-212}{360} \qquad = -\ 23.6$ Vert. Comp. in U_3U_4 $=\frac{15}{60}\left(23.6\times 8+\frac{480}{30}\right)\frac{30}{75} \qquad = -\ 20.5$ Vert. Comp. in M_3U_4 $=\frac{480}{30}\times\frac{30}{75}\times\frac{45}{30} \qquad = +\ 9.6$ Vert. Comp. in M_3L_4 in tons for E_{40} $=23.6+20.5-9.6 \qquad = -\ 34.5$ Compression for $E_{50}=34.5\times 1.414\times\frac{5}{4}\times 2 \qquad = -121.9$ This is so much smaller than the dead tension that compression will never actually occur in this bar.
U_4M_5	3 at L_5 Max. Tension.	Shear in panel 4–5 $=\frac{16{,}364+(284+114)114}{360}-\frac{230}{30}$ $=171.5-7.7 \qquad = +163.8$ Tension for $E_{50}=\ 163.8\times 1.280\times\frac{5}{4}\times 2 \qquad = +524.2$

MAXIMUM LIVE STRESSES IN MAIN DIAGONALS—*Continued*

Bar.	Position of Loads.	Computations.	
M_5L_6	5 at L_6 Max. Tension.	Shear in panel 5–6 $=\frac{16{,}364+(284+94)94}{360}-\frac{830}{30}$	
		$=144.1-27.7$	$=+116.4$
		Vert. Comp. in M_5U_6 $=\frac{27.7}{2}$	$=+\ 13.8$
		Vert. Comp. in M_5L_6 in tons for E_{40} $=116.4+13.8$	$=+130.2$
		Tension for $E_{50}=\ 130.2\times1.280\times\frac{5}{4}\times2$	$=+416.6$
M_7L_8	4 at L_8 Max. Tension.	Shear in panel 7–8 $=\frac{16{,}364+(284+29)29}{360}-\frac{480}{30}$	
		$=70.7-16.0$	$=+\ 54.7$
		Vert. Comp. in M_7U_8	$=+\ \ 8.0$
		Vert. Comp. in M_7L_8 in tons for E_{40} $=54.7+8.0$	$=+\ 62.7$
		Tension for $E_{50}=\ 62.7\times1.280\times\frac{5}{4}\times2$	$=+200.6$

MAXIMUM LIVE STRESSES IN INCLINED END-POSTS, CHORDS, AND MAIN VERTICALS IN UNITS OF 1000 POUNDS

This table shows all necessary computations.

Bar.	Position of Loads.	Computations.		
L_0M_1	4 at L_1	Shear in panel 0–1 $\frac{16{,}364+(284+239)239}{360}-\frac{480}{30}$ $=392.7-16.0$	$=$	376.7
		Compression in L_0M_1 for E_{60} $=376.7\times1.414\times\frac{5}{4}\times2$	$=$	1331.6
L_0L_1 L_1L_2	4 at L_1	Tension for $E_{60}=376.7\times\frac{5}{4}\times2$	$=$	941.8
M_1U_2	8 at L_2	Moment at L_2 $=\frac{16{,}364+(284+234)234}{6}-2851$	$=$	20,078
		Hor. Comp. in M_1U_2 in tons for E_{40} $=\frac{20{,}078}{60}=334.6$		
		Compression for $E_{60}=334.6\times1.414\times\frac{5}{4}\times2$	$=$	1182.8
L_2L_3 L_3L_4	8 at L_2	Tension for E_{60} $=$ $334.6\times\frac{5}{4}\times2$	$=$	836.5
L_2U_2	5 at L_2. Loads coming on from left.	Panel load at L_2 $10\times\frac{7}{30}+80\times\frac{22.5}{30}+52\times\frac{13}{30}$	$=$	84.9
		Panel load at L_1 $52\times\frac{17}{30}+80\times\frac{11.5}{30}+10\times\frac{27}{30}$	$=$	69.1
		Tension for E_{60} $=\left(84.9+\frac{69.1}{2}\right)\frac{5}{4}\times2$	$=$	298.5
U_2U_3 and U_3U_4	12 at L_3	Moment about L_4 of left reaction $\frac{16{,}364+(284+230)230}{3}$	$=$	44,861
		Moment about L_4 of loads to left of section $2155+116\times62+\frac{230}{30}\times60$ $+26\times\frac{23.5}{30}\times60$	$=$	11,029
		Hor. Comp. in bar in tons for E_{40} $=\frac{44{,}861-11{,}029}{75}=451.1$ tons.		
		Compression for $E_{60}=451.1\times1.031\times\frac{5}{4}\times2$	$=$	1162.7

MAXIMUM LIVE STRESSES IN INCLINED END-POSTS, CHORDS, AND MAIN VERTICALS—*Continued*

Bar.	Position of Loads.	Computations.
U_4L_4	2 at L_5 Max. Compression.	Moment of left reaction about 6 $=\dfrac{16{,}364+(284+109)109}{12p}\times 6p = 29{,}600$ Panel load at $L_4=\dfrac{80}{30} = 2.67$ Compression in bar for E_{50} $=(\dfrac{29{,}600}{300}-2.67)\dfrac{5}{4}\times 2 = 240.0$
U_4L_4	3 at L_4 Max. Tension Loads coming on from left.	Moment about U_4 of all forces to right of section through panel 3–4 $=\dfrac{8}{12}[16{,}364+(284+24)24]-230 = 15{,}607$ Hor. Comp. (Bar $L_3L_4+M_3L_4$) $=\dfrac{15{,}607}{75} = 208.1$ Moment about U_2 of loads to right of section through panel 2–3 $=\dfrac{10}{12}[16{,}364+(284+24)24]$ $-(7668-192)=19{,}797-7476 = 12{,}321$ Stress in L_2L_3 in tons for $E_{40}=\dfrac{12{,}321}{60} = 205.3$ Hor. Comp. in tons for E_{40} in M_3L_4 = Vert. Comp. = 2.8 tension. Panel load at L_4 (Load 3 at L_4) $=50-\dfrac{230}{30}+20\left(\dfrac{25+20}{30}\right)+\dfrac{13(11+6)}{30}$ $=79.7$ Tension in U_4L_4 for $E_{50}=2\times\dfrac{5}{4}\times(79.7-2.8) = 192.2$
* U_4U_5 and U_5U_6	15 at L_6	Moment about L_6 of left reaction $=\dfrac{16{,}364+(284+189)189}{2} = 52{,}880$ Moment about L_6 of loads to left of section $=4632+152\times 62+\dfrac{80\times 16.5}{30}\times 60 = 16{,}696$ Compression in bar for E_{50} $=\dfrac{52{,}880-16{,}696}{75}\times\dfrac{5}{4}\times 2 = 1206.1$
L_4L_5 and L_5L_6	15 at L_4	Moment about U_4 $=\dfrac{16{,}364+(284+219)219}{3}-10{,}816 = 31{,}358$ Tension in bar for E_{50} $=\dfrac{31{,}358}{75}\times\dfrac{5}{4}\times 2 = 1045.3$
U_6L_6	3 at L_7	Shear in panel 6–7 $=\dfrac{16{,}364+(284+54)54}{360}-\dfrac{230}{30}$ $=96.2-7.7 = 88.5$ Stress in $M_5U_6=0$ Compression in bar $=88.5\times 2\times\dfrac{5}{4} =$ †221.2

* Note that shear for this loading in panel 4–5 is positive, hence counter M_5L_4 is not in action.
† Note that this is larger than maximum panel load, hence is maximum stress.

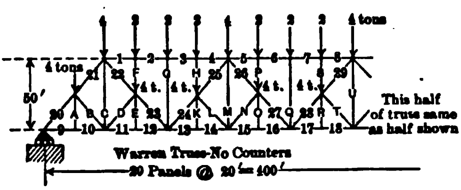
MAXIMUM LIVE STRESSES IN SECONDARY MEMBERS, IN UNITS OF 1000 POUNDS

This table shows all necessary computations.

Bar.	Position of Loads.	Computations.
M_1L_1 M_3L_3 M_5L_5	13 at L_1, L_3 or L_5	It has been previously determined that a maximum occurs with either load (12), (13), or (14) at a secondary panel point, hence panel loading for each case is computed below. Load (12) $\frac{13(4+9+11+6)}{30}+10\times\frac{17}{30}+20\frac{25+30+25+20}{30} = 85.4$ Load (13) $\frac{13(4+16+11+5)}{30}+10\times\frac{12}{30}+20\left(\frac{100}{30}\right) = 86.3$ Load (14) $\frac{13(21+16+10+5)}{30}+10\times\frac{7}{30}+20\frac{(15+20+25+30)}{30} = 84.8$ Tension in bar for $E_{50}=86.3\times\frac{5}{4}\times 2 = 215.7$
M_1L_2	13 at L_1	Compression in bar for $E_{50}=86.3\times\frac{5}{4}\times 1.414 = 152.5$
M_3U_4	13 at L_3	Tension in bar for $E_{50}=86.3\times\frac{5}{4}\times\frac{45}{75}\times 2\times 1.202 = 155.6$
M_5U_6	13 at L_5	Tension in bar for $E_{50}=86.3\times\frac{5}{4}\times 1.280 = 138.1$

PROBLEMS

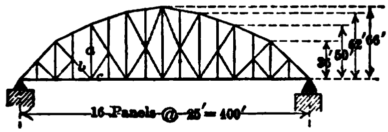
50. Uniform live load, 2000 lbs. per foot on bottom chord.
Locomotive excess, 20,000 lbs. " "
Dead load, 800 lbs. per foot on bottom chord.
Dead load on top chord and intermediate panel points as shown in figure.



PROB. 50.

Compute maximum stress in each member, following rules given in previous problems as to arrangement of computations, using special care to number and letter the bars exactly as in figure.

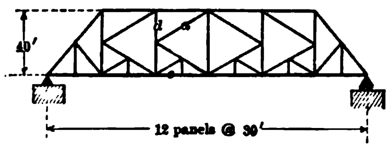
51. Uniform live load, 2000 lbs. per foot on bottom chord.
Locomotive excess, 20,000 lbs.
Dead load, 1200 lbs. per foot on bottom chord.
Dead load, 600 lbs. per foot on top chord.



PROB. 51.

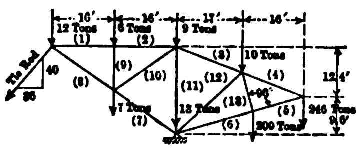
a. Draw influence line for stress in bar *a* and compute its maximum value for above loading.
b. Draw influence line for stress in bar *b* and compute its maximum value.
c. Draw influence line for stress in bar *c* and compute its maximum value.

52. Draw influence line for stress in bar *a* of trusses shown in Prob. 41.
a. Truss I. Truss has 12 panels at 25 ft. and height of 60 ft.
b. Truss J. Truss has 8 panels at 20 ft. and height of 30 ft.



PROB. 53.

53. Dead load, top chord, 2250 lbs. per ft. per truss = 135,000 lbs. per panel point (approx.).
Dead load, bottom chord, 3500 lbs. per ft. per truss = 105,000 lbs. per panel point.
Uniform live load, bottom chord, 3000 lbs. per ft. per truss.
Draw influence lines for stresses in *a*, *b*, *c* and *d* and compute maximum values of live stresses.



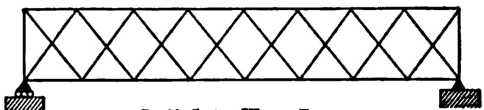
PROB. 53½.

53½*a*. Compute stresses in bars (10) and (12) by method of moments.
b. Compute stress in bar (11) by method of joints.

CHAPTER VIII

TRUSSES WITH MULTIPLE WEB SYSTEMS, LATERAL AND PORTAL BRACING, TRANSVERSE BENTS, VIADUCT TOWERS

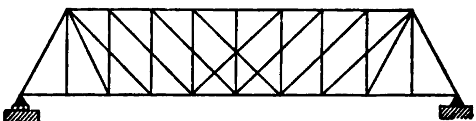
110. Trusses with Multiple Web Systems. Trusses of this type are statically undetermined, but are frequently built for spans of moderate length, as many engineers believe that more rigidity is thereby obtained. The trusses shown in Figs. 169 and 170 represent the more common types of such structures.



Double System Warren Truss
This truss should always have an even number of panels.

FIG. 169.

The fact that such trusses are indeterminate makes it impossible to correctly determine the stresses by methods previously given. Methods of accurately computing such stresses will be given later in full; but it may be said here that these methods can only be applied to trusses in which the



Whipple Truss

FIG. 170.

areas of the various members are known or assumed in advance, hence, if used in design they must be applied through a series of approximations, the areas being first determined approximately, the stresses then computed, and the areas revised if necessary, this process being continued until a sufficiently accurate design is finally obtained. The accuracy of the approximate method ordinarily employed for such trusses is, however, sufficiently

high to make unnecessary the employment of more exact methods for the simple types of trusses shown in this article.

The approximate method in common use consists of the separation of the web members into systems, each of which is considered to be entirely distinct from the others. This amounts in reality to dividing the truss into two or more separate trusses with common top and bottom chords. The maximum web stresses in each of these trusses may then be computed in the ordinary manner, assuming each system to carry only such panel loads as are applied to it. The chord stress in any bar corresponding to each web system may then be computed the total stress in any chord bar being the sum of the stresses determined for each system. The method of determining stresses is clearly illustrated by the example which follows:

111. Approximate Determination of Maximum Stresses in a Double System Warren Truss.

Problem. Let the problem be the determination of the maximum stresses in all the bars of the truss shown in Fig. 171, with the following loads:

Dead weight of bridge,
800 lbs. per ft. per truss, top chord =12,800 lbs. per panel.
400 " " " " " bottom " = 6,400 " " "
Uniform live load,
3000 lbs. per ft. per truss, top chord =48,000 lbs. per panel.
Locomotive excess, =40,000 lbs.

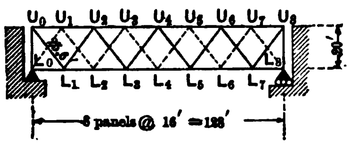


FIG. 171.

The two web systems into which this truss is assumed to be divided are shown by the full and dotted lines respectively. The number of redundant bars in the structure may be determined in the usual manner by comparing the total number of bars with twice the number of joints less three. This comparison shows that the number of bars is one in excess of the number needed for statical determination.

Index Stresses. These may be written for each web system separately in the ordinary manner, considering the full sys-

tem to carry only such loads as act at even numbered top chord panel points, and the dotted system to carry all other panel loads. With the web index stresses known, the chord stresses may be written in the ordinary manner, by adding the diagonal stresses at each joint successively, both systems being considered. Fig. 172 shows the index stresses for one half the truss.

Were this truss to have an odd number of panels it would be necessary to write the index stresses for the web members in both halves of the truss, since neither system would be symmetrical.

The index stresses were written as usual by beginning at the centre of the truss. The left reaction $=4\frac{1}{2}(6.4)+3\frac{1}{2}(12.8)=73.6$,

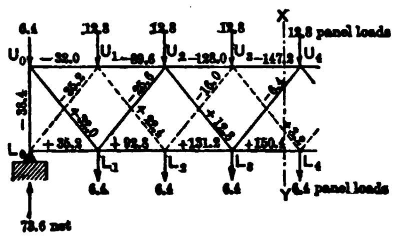


FIG. 172.

which checks the web index stresses. The chord index stresses may be checked by the method of moments as in the ordinary truss, provided due allowance is made for the stress in the diagonal cut by the section selected. In this case the stress in the centre panel of the bottom chord may be checked by computing the moments about U_4 of the external forces to left of section XY, and subtracting from it the moment of the stresses in diagonal U_3L_4, making use of the fact that the moment about U_4 of the stress in this diagonal equals the product of its vertical component, i.e., its index stress, and the panel length.

The stress in L_3L_4 as determined from the index stresses $=150{,}400\times\frac{16}{20}=120{,}300$ lbs. By the method of moments the stress in $L_3L_4=\frac{1}{8}\cdot 1200\cdot\frac{128\cdot 128}{20}-\frac{3200\times 16}{20}=120{,}300$ lbs.

This value agrees with that obtained from the index stresses, and consequently shows the correctness of these stresses. The

actual dead stresses may be computed from the index stresses in the usual manner and will not be given.

Maximum Live Web Stresses. To determine the maximum live web stresses consider each system as an independent truss, and determine the stresses in the usual manner by the method of shear. The panel loads will be those corresponding to the panel lengths of the actual truss.

MAXIMUM LIVE WEB STRESSES IN UNITS OF 1000 POUNDS

Bar.	Truss System.	Uniform Load at Panel Points.	Loco. Excess at Panel Points.	Vert. Comp. in Bars.	$\frac{L}{h}$	Stress.
U_0L_0	Full	U_0-U_2-U_4-U_6	U_0	$\frac{12}{8}48+24+40=136.0$	1.00	−136.0
U_0L_1	Full	U_2-U_4-U_6	U_2	$\frac{12}{8}48+\frac{6}{8}40=102.0$	1.28	+130.6
L_1U_2	Full	U_2-U_4-U_6	U_2	$\frac{12}{8}48+\frac{6}{8}40=102.0$	1.28	−130.6
U_2L_3	Full	U_4-U_6	U_4	$\frac{6}{8}48+\frac{4}{8}40=56.0$	1.28	+ 71.7
L_3U_4	Full	U_4-U_6	U_4	$\frac{6}{8}48+\frac{4}{8}40=56.0$	1.28	− 71.7
U_4L_5	Full	U_6	U_6	$\frac{2}{8}48+\frac{2}{8}40=22.0$	1.28	+ 28.2
L_5U_6	Full	U_6	U_6	$\frac{2}{8}48+\frac{2}{8}40=22.0$	1.28	− 28.2
L_0U_1	Dotted	U_1-U_3-U_5-U_7	U_1	$\frac{16}{8}48+\frac{7}{8}40=131.0$	1.28	−167.7
U_1L_2	Dotted	U_3-U_5-U_7	U_3	$\frac{9}{8}48+\frac{5}{8}40=79.0$	1.28	+101.1
L_2U_3	Dotted	U_3-U_5-U_7	U_3	$\frac{9}{8}48+\frac{5}{8}40=79.0$	1.28	−101.1
U_3L_4	Dotted	U_5-U_7	U_5	$\frac{4}{8}48+\frac{3}{8}40=39.0$	1.28	+ 49.9
L_4U_5	Dotted	U_5-U_7	U_5	$\frac{4}{8}48+\frac{3}{8}40=39.0$	1.28	− 49.9
U_5L_6	Dotted	U_7	U_7	$\frac{1}{8}48+\frac{1}{8}40=11.0$	1.28	+ 14.1

As the truss is a Warren truss no counters are needed, but the maximum stress of both kinds should be computed in all bars in which reversal of stress may occur, since the area of such bars is dependent upon the magnitude of both kinds of stresses.

Maximum Live Chord Stresses. For the maximum stresses due to the uniform live load, the index stresses should be written and the maximum stresses computed in the ordinary manner. It should be observed that for this truss the live stresses cannot be obtained from the dead stresses by multiplying by the ratio between the two loads since the live stress is not distributed in the same manner between the top and bottom chord.

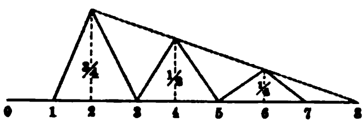
To determine the maximum stresses due to the locomotive excess it is necessary to decide in which system the bar should be considered in order that the stress may have its maximum value. This can usually be settled by inspection, but if doubt exists the maximum stresses for both systems should be written and the larger value used.

The following table gives the maximum stresses due to locomotive excess in all bars:

Bar.	System.	Load at	Stress.
U_0U_1	Full	U_2	$\frac{6}{8}40\times\frac{16}{20}=-24$
U_1U_2	Dotted	U_3	$\frac{5}{8}40\times\frac{32}{20}=-40$
U_2U_3	Full	U_4	$\frac{4}{8}40\times\frac{48}{20}=-48$
U_3U_4	Dotted	U_5	$\frac{3}{8}40\times\frac{64}{20}=-48$
L_0L_1	Dotted	U_1	$\frac{7}{8}40\times\frac{16}{20}=+28$
L_1L_2	Full	U_2	$\frac{6}{8}40\times\frac{32}{20}=+48$
L_2L_3	Dotted	U_3	$\frac{5}{8}40\times\frac{48}{20}=+60$
L_3L_4	Full	U_4	$\frac{4}{8}40\times\frac{64}{20}=+64$

Concentrated Load System. The position of loads for the maximum stresses in this truss due to a concentrated load system may be determined by the use of influence lines. A complete solution for all bars will not be given, but the typical example which follows includes all the important points which are likely to arise.

Bar U_0L_1 Position of Loads for Maximum Stress for Cooper's E_{40}. The influence line for the vertical component in this case is shown in Fig. 173 and indicates that heavy loads should lie in panels 1–2 and 2–3, with one of the loads at point 2. The method of moving up the loads, making use if necessary of the tangents of the angles between the influence lines and the horizontal, will enable us to determine which load should lie at panel point 2. As the loads in panels 1–2 and 2–3 will be of the most importance in deciding this question, it is advisable to first determine the position, considering only the loads in these two panels, and then investigate to see whether a change in position will diminish or increase the stress. Since the influence line for these two panels is composed of two straight lines, the loads in these panels should be placed so as to give maximum moment at the centre of a 32-ft. span. It is evident from inspection that this occurs with load (3), at panel point 2. With load (3) just to left of panel point 2 the total load on the left panel of each of the other two-panel segments is greater than that on the right, and movement to the left until load (4) comes to panel point 2 will not change this relation, hence it is evident that load (4), at panel point 2, gives a smaller stress than load (3). Movement to the right until load 2 is at the panel point will decrease materially the stress due to loads in panels 1–2 and 2–3, but will increase the effect of the loads in the other panels. This will probably decrease the stress in the bar, but as the effect of this change cannot be so readily determined by inspection as in the other case, both cases will be computed, as this is simpler than to attempt to determine the exact change by the process of moving up the loads.



Influence Line for Vertical Component in Bar U_0L_1

FIG. 173.

Vertical component of stress in bar U_0L_1. Load (2) at U_2:

Load at panel point 2. $10\times\frac{8}{16}+20\left(\frac{16+11+6+1}{16}\right)=47.5;$

Load at panel point 4. $13\times\frac{8+13+13+8}{16}=34.1;$

Load at panel point 6. $20\times\frac{8+13+14+9}{16}=55.0;$

V.C. in bar from influence line ordinate

$$47.5\times\frac{3}{4}+34.1\times\frac{1}{2}+55\times\frac{1}{4}=66.4;$$

Load (3) *at* U_2:

Load at panel point 2. $10\times\frac{3}{16}+20\times\frac{11+16+11+6}{16}=56.9;$

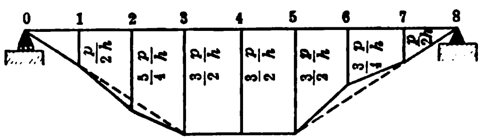
Load at panel point 4. $13\times\frac{3+8+14+13}{16}+10\times\frac{5}{16}=34.0;$

Load at panel point 6. $20\times\frac{3+8+13+14}{16}+13\times\frac{5}{16}=51.6;$

V.C. in bar $56.9\times\frac{3}{4}+34.0\times\frac{1}{2}+51.6\times\frac{1}{4}=72.6;$

This latter value is the maximum and should be used in the design. The position of load for the other web members may be determined in a similar manner.

Bar U_3U_4. *Position of Loads for Maximum Stress for Coopers* E_{40}. The influence line for this bar is shown by full lines



Influence Line- Stress in Bar U_3U_4

FIG. 174.

in Fig. 174. The values of the ordinates are given by the following computations, the bar in question being considered as a part of the dotted system for loads at odd-numbered panel points, and as a part of the full-line system for loads at other panel points.

Load at 7—Bar in dotted system—Ordinate $=\frac{1}{8}\times 4\frac{p}{h}=\frac{p}{2h}$.

Load at 6—Bar in full system—Ordinate $=\frac{2}{8}\times 3\frac{p}{h}=\frac{3p}{4h}$.

Load at 5—Bar in dotted system—Ordinate $=\frac{3}{8}\times 4\frac{p}{h}=\frac{3p}{2h}$.

Load at 4—Bar in full system—Ordinate $=\frac{4}{8}\times 3\frac{p}{h}=\frac{3p}{2h}$.

Load at 3—Bar in dotted system—Ordinate $=\frac{3}{8}\times 4\frac{p}{h}=\frac{3p}{2h}$.

Load at 2—Bar in full system—Ordinate $=\frac{2}{8}\times 5\frac{p}{h}=\frac{5p}{4h}$.

Load at 1—Bar in dotted system—Ordinate $=\frac{1}{8}\times 4\frac{p}{h}=\frac{p}{2h}$.

Inspection shows that for this case the moment will certainly increase as the loads come on from the right until load (6) reaches panel point 3. As the loads move still further it is more difficult to determine exactly the position for maximum moment. An approximate determination based upon the assumption that the sloping influence lines coincide with the dotted lines may be used, the error thus introduced being comparatively small. Assuming this condition, the position for maximum moment will occur with the load on panels 5 to 8 inclusive, equal to that on panels 1 to 3 inclusive.

Try load (6) to left of 3:

Load on 1–3 = 103; load on 5–8 = 118. $\therefore$ move up (7).

Try load (7) to left of 3:

Load on 1–3 = 116; load on 5–8 = 108. $\therefore$ load (7) at panel point 3 will probably give the maximum value.

It should be noticed that for this position load (12) is at panel point 5.

To determine the stress for this position compute the panel loads at panel points 1, 2, 6 and 7. Compute also the panel loads at 3 and 5 due to loads in panels 2–3 and 5–6. Multiply each of these panel point loads by the corresponding ordinate to the influence line, and multiply the loads in panels 3–4 and

4–5 by the influence line ordinate in these panels. The summation of these quantities gives the stress in the bar.

Stress in U_3U_4. Load (7) at panel point 3:

Load at panel point 1. $10\times\frac{11}{p}+20\times\frac{13+8+3}{p} \qquad =\frac{590}{p}$

Load at panel point 2. $20\times\frac{3+8+13+14}{p}+13\times\frac{5}{p} \qquad =\frac{825}{p}$

Load at panel point 3, (loads in panel 2–3 only).

$$20\times\frac{2}{p}+\frac{13\times 11}{p} \qquad =\frac{183}{p}$$

Load at panel point 5, (loads in panel 5–6 only).

$$20\times\frac{11+6}{p} \qquad =\frac{340}{p}$$

Load at panel point 6. $20\times\frac{5+10}{p}+13\frac{13+8+2}{p} \qquad =\frac{599}{p}$

Load at panel point 7. $13\times\frac{3+8+14}{p}+\frac{13\times 13}{p}+\frac{16\times 4}{p}=\frac{558}{p}$

$$\text{Stress in bar}=\left(\frac{590+558}{p}\right)\frac{p}{2h}+\frac{825}{p}\cdot\frac{5p}{4h}+\frac{599}{p}\cdot\frac{3p}{4h}$$

$$+\left(\frac{183+340+89\times 16}{p}\right)\frac{3p}{2h}=\frac{4975}{20}=-248.7$$

As the method used for determining the position in this case was not a rigid one, the stress in the bar with load (13) at panel point 5 will be computed for comparison:

Load at panel point 1. $10\times\frac{6}{p}+20\times\frac{14+13+8+3}{p}=\frac{820}{p}$

Load at panel point 2. $20\times\frac{3+8+13}{p}+1\times 3\frac{10+5}{p}=\frac{675}{p}$

Load at panel point 3, (loads in panel 2–3 only).

$$13\times\frac{6+11}{p} \qquad =\frac{221}{p}$$

Load at panel point 5, (loads in panel 5–6 only).

$$20\times\frac{11}{p}+13\times\frac{2}{p} \qquad =\frac{246}{p}$$

Load at panel point 6. $20\times\frac{5}{p}+13\times\frac{14+13+7+2}{p}=\frac{568}{p}$

Load at panel point 7. $13\times\frac{3+9+14}{p}+2\times\frac{13\times 6.5}{p}=\frac{507}{p}$

$$\text{Stress in bar}=\left(\frac{820+507}{p}\right)\frac{p}{2h}+\frac{675}{p}\cdot\frac{5p}{4h}+\frac{568}{p}\cdot\frac{3p}{4h}$$
$$+\left(\frac{221+246+96\times 16}{p}\right)\frac{3p}{2h}=\frac{4938}{20}=-246.9,$$

or considerably less than the value previously obtained.

112. Approximate Determination of Maximum Stresses in a Whipple Truss. The Whipple truss shown in Fig. 170 may be treated in a similar manner to the double-system Warren truss. The two systems into which the truss may be divided are shown in Fig. 175 by dotted and full lines, respectively.

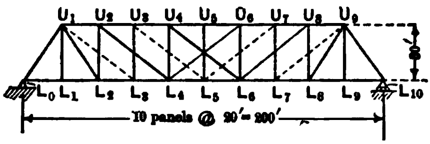


FIG. 175.

This truss has one redundant member assuming that only one of centre diagonals of full system can act at once and the removal of any one of the web members except the end diagonals or end verticals would make the truss statically determined. This truss has, however, one element of uncertainty which does not exist in the double-system Warren truss previously treated, viz., that the end verticals U_1L_1 and U_9L_9 do not distinctly belong to either system. This ambiguity is troublesome in determining how to place the live load for maximum stresses. The usual solution in this case is to use these verticals in such a manner as to give the maximum stress in the bar under consideration. For example, if the problem be the determination of the maximum tension in bar U_2L_4, the bar L_9U_9 should be considered as a part of the full system, and the bar U_1L_1 as a part of the dotted system and the truss loaded accordingly. The following example illustrates the method of solution for such a truss:

Problem. Let the problem be the determination of the maximum stresses in all the bars of the truss shown in Fig. 176.

Dead weight of bridge,

1200 lbs. per ft. per truss, bottom chord = 24,000 lbs. per panel.
600 " " " " " top " = 12,000 " " "

Uniform live load,

3000 lbs. per ft. per truss, bottom chord = 60,000 lbs. per panel.
Locomotive excess, = 40,000 lbs.

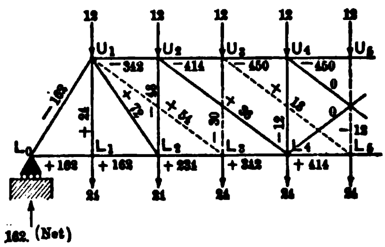


FIG. 176.

Index Stresses. The index stresses may be written for the dotted system by beginning at the centre, the bar U_3L_5 carrying one-half of the centre panel loads, the dotted system being symmetrical, and panel point 5 at its centre. For the full system the shear in the centre panel is zero, and the stresses in bars U_4L_6 and L_4U_6 will each be considered as zero. It should be noticed that above conditions are based on the assumption that the dead panel load at both L_1 and L_9 is equally divided between the two truss systems. The index stresses present no special difficulty. The only point to which attention should be called is the necessity for correcting the index stresses in the diagonals in the same manner as in the inclined chord trusses previously considered.

In this problem the diagonal index stresses are corrected to conform to the slope of the diagonal U_1L_2; i.e., the stresses in the other diagonals are each doubled before the chord index stresses are written:

Check calculations,

$$\text{Stress in } U_4U_5 \text{ by method of moments} = \frac{1}{8}\cdot\frac{1800\times200\times200}{30}$$
$$= 300{,}000.$$

Stress in U_4U_5 in 1000 lb. units from index stresses $= 450\times\frac{20}{30} = 300.$

The actual dead stresses are given in the following table in which the column headed ratio gives for each web member its length divided by its vertical projection; and for each chord member the fraction $\frac{2}{3}$, which equals the horizontal projection of the diagonal U_1L_2 divided by its vertical projection.

DEAD STRESSES IN UNITS OF 1000 POUNDS

Bar.	Index Stress.	Ratio.	Dead Stress.	Bar.	Index Stress.	Ratio.	Dead Stress.
L_0U_1	−162	1.201	−194.6	L_0L_1	+162	$\frac{2}{3}$	+108
U_1L_2	+ 72	1.201	+ 86.5	L_1L_2	+162	$\frac{2}{3}$	+108
U_1L_1	+ 24	1.000	+ 24.0	L_2L_3	+234	$\frac{2}{3}$	+156
U_2L_2	− 48	1.000	− 48.0	L_3L_4	+342	$\frac{2}{3}$	+228
U_3L_3	− 30	1.000	− 30.0	L_4L_5	+414	$\frac{2}{3}$	+276
U_4L_4	− 12	1.000	− 12.0	U_1U_2	−342	$\frac{2}{3}$	−228
U_5L_5	− 12	1.000	− 12.0	U_2U_3	−414	$\frac{2}{3}$	−276
U_1L_3	+ 54	$\frac{5}{3}$	+ 90.0	U_3U_4	−450	$\frac{2}{3}$	−300
U_2L_4	+ 36	$\frac{5}{3}$	+ 60.0	U_4U_5	−450	$\frac{2}{3}$	−300
U_3L_5	+ 18	$\frac{5}{3}$	+ 30.0				
U_4L_6	0	$\frac{5}{3}$	0				

Before computing the live stresses the necessity for counters will be investigated. To do this consider each system separately.

Maximum live compression in U_3L_5—load L_1 and L_3-E at L_3 $V.C.=\frac{4}{10}\ 60+\frac{3}{10}\ 40$. This is considerably larger than the corresponding figure for dead tension, hence a counter L_3U_5 is required.

Maximum live compression in U_2L_4—load L_1 and L_2-E at L_2 $V.C.=\frac{3}{10}\,60+\frac{2}{10}\,40=26$. This with impact added would be larger than the corresponding figure for dead tension, hence a counter L_2U_4 should be used.

LIVE WEB STRESSES IN UNITS OF 1000 POUNDS

This table shows all necessary computations.

Bar.	Uniform Load at Panel Points.	E at Panel Point.	Vertical Component of Maximum Stress.	Ratio.	Stress.
L_0U_1	L_1 to L_9 incl.	L_1	$60\times4\frac{1}{2}+\frac{9}{10}40=306$	1.201	-367.5
U_1L_2	L_2, L_4, L_6, L_8, L_9	L_2	$\frac{21}{10}60+\frac{8}{10}40=158$	1.201	$+189.8$
U_1L_1	L_1	L_1	$60+40 \quad =100$	1.000	$+100.0$
U_2L_4	L_4, L_6, L_8, L_9	L_4	$\frac{13}{10}60+\frac{6}{10}40=102$	$\frac{5}{3}$	$+170.0$
U_2L_2	L_4, L_6, L_8, L_9	L_4	$\frac{13}{10}60+\frac{6}{10}40=102$	1.000	-102.0
U_4L_6	L_6, L_8, L_9	L_6	$\frac{7}{10}60+\frac{4}{10}40=\ 58$	$\frac{5}{3}$	$+\ 96.7$
U_4L_4	L_6, L_8, L_9	L_6	$\frac{7}{10}60+\frac{4}{10}40=\ 58$	1.000	$-\ 58.0$
U_6L_8	L_8, L_9	L_8	$\frac{3}{10}60+\frac{2}{10}40=\ 26$	$\frac{5}{3}$	$+\ 43.3$
U_1L_3	L_3, L_5, L_7, L_9	L_3	$\frac{16}{10}60+\frac{7}{10}40=124$	$\frac{5}{3}$	$+206.7$
U_3L_5	L_5, L_7, L_9	L_5	$\frac{9}{10}60+\frac{5}{10}40=\ 74$	$\frac{5}{3}$	$+123.3$
U_3L_3	L_5, L_7, L_9	L_5	$\frac{9}{10}60+\frac{5}{10}40=\ 74$	1.000	$-\ 74.0$
U_5L_7	L_7, L_9	L_7	$\frac{4}{10}60+\frac{3}{10}40=\ 36$	$\frac{5}{3}$	$+\ 60.0$
U_5L_5	L_7, L_9	L_7	$\frac{4}{10}60+\frac{3}{10}40=\ 36$	1.000	$-\ 36.0$

LIVE CHORD STRESSES IN UNITS OF 1000 POUNDS

This table shows all necessary computations.

Bar.	Live Stress Due to Uniform Load $=\frac{30}{18}$ of Dead Stress.	Position of E.	Stress Due to E.	Total Maximum Live Stress.
L_0L_1	$108\times\frac{30}{18}=+180$	L_1	$\frac{9}{10}40\times\frac{2}{3}=+24.0$	$+204.0$
L_1L_2	$108\times\frac{30}{18}=+180$	L_1	$\frac{9}{10}40\times\frac{2}{3}=+24.0$	$+204.0$
L_2L_3	$156\times\frac{30}{18}=+260$	L_2	$\frac{8}{10}40\times\frac{4}{3}=+42.7$	$+302.7$
L_3L_4	$228\times\frac{30}{18}=+380$	L_3	$\frac{7}{10}40\times\frac{6}{3}=+56.0$	$+436.0$
L_4L_5	$276\times\frac{30}{18}=+460$	L_3	$\frac{7}{10}40\times\frac{6}{3}=+56.0$	$+516.0$
U_1U_2	$228\times\frac{30}{18}=-380$	L_3	$\frac{7}{10}40\times\frac{6}{3}=-56.0$	-436.0
U_2U_3	$276\times\frac{30}{18}=-460$	L_4	$\frac{6}{10}40\times\frac{8}{3}=-64.0$	-524.0
U_3U_4	$300\times\frac{30}{18}=-500$	L_5	$\frac{5}{10}40\times\frac{10}{3}=-66.7$	-567.7
U_4U_5	$300\times\frac{30}{18}=-500$	L_5	$\frac{5}{10}40\times\frac{10}{3}=-66.7$	-567.7

The determination of the maximum stresses in a Whipple truss for a concentrated load system should be made in a manner similar to that employed for the Warren truss, making use of influence lines to determine the position of loads. Computations for such loads will be omitted as involving no new methods.

113. Skew Bridges. It is often necessary to construct bridges the abutments or piers of which are not at right angles to the bridge axis. Plans of such bridges are shown in Figs. 177 and 178.

In structures of this sort the trusses are frequently unsymmetrical, as is evidently the case for the trusses shown in Fig. 177. The trusses shown in Fig. 178 are symmetrical, but the panel loads are affected somewhat by the skew of the ends. If it is desired to use inclined end diagonals for such trusses, they should both

have the same inclination to the horizontal in order that the end portal may lie in a plane. For simplicity in construction the floor beams should be located at right angles to the trusses. In order to satisfy both of these conditions it is frequently desirable to place the end hangers at an inclination to the vertical, as shown in Fig. 179.

The computation of stresses in such trusses may be made in the same manner as in the trusses already considered, and requires no special treatment. If difficulties occur in determining the position of the loads, they may usually be solved by using the influence line.

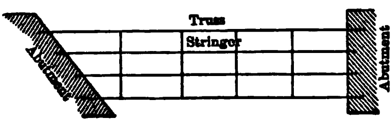


FIG. 177.

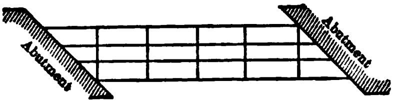


FIG. 178.

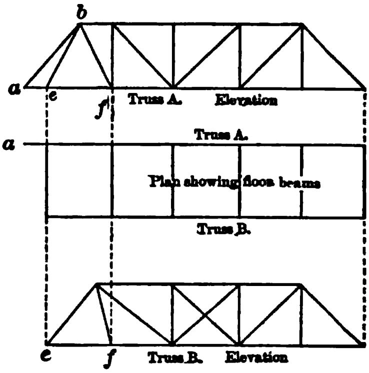


FIG. 179.

114. Lateral and Portal Bracing. It is evident that a bridge in which the floor beams form the only connection between the trusses would be unstable laterally, especially if of long span. This instability would be due partially to its inability to withstand the force of the wind acting upon the truss itself and upon the train or other live load which may be upon the bridge, and partially to the lateral vibration to which it may be subjected by the live load, this being especially severe for railroad bridges exposed to swift and heavy train service. In addition to the insecurity of such a structure as a whole another disadvantage would be the fact that the top chords would have to be made much heavier than would be the case were they to be rigidly braced, since they would be in the condition of very long columns unsupported laterally, and the extra material used to give them

sufficient strength would, in most cases, be more than sufficient to provide for lateral bracing.

For these reasons it is considered necessary to use lateral bracing in all bridges. In through bridges this bracing should consist of a horizontal truss in the plane of the bottom chord, another in the plane of the top chord when the depth permits (trusses of insufficient depth to permit the use of overhead bracing are called pony trusses and should be avoided), and vertical bracing between the verticals of as great a depth as the allowable clearance permits.[1] In deck bridges a horizontal truss may be used in the plane of the upper chord and vertical sway bracing

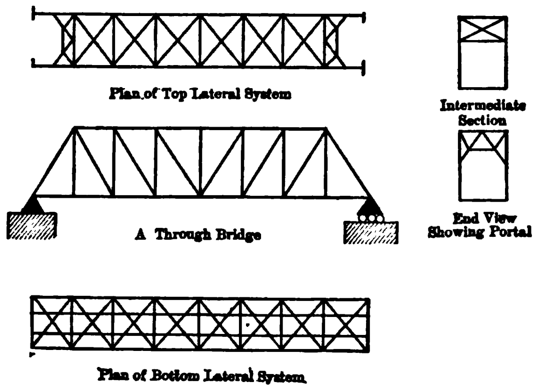


FIG. 180.

between the vertical members, no horizontal bracing being used in the plane of the bottom chord, or all three systems of bracing may be used.

In through bridges the end reactions of the top lateral truss cannot be directly transmitted to the abutments owing to the necessity of preserving a suitable opening for the traffic, hence portal bracing is required in the plane of the end posts, the purpose of this bracing being to tie the end posts together and make thereby a rigid frame by which the end reactions can be transferred to the abutments.

Figs. 180 and 181 show the lateral bracing in through and deck bridges respectively.

[1] One of the reasons for using vertical bracing when both top and bottom lateral systems are used is to assist in distributing unequal train loads between the trusses.

115. Lateral-bracing Trusses. Lateral trusses may be either statically determinate, or statically indeterminate, according to whether the diagonals are tension rods, or riveted members capable of carrying both tension and compression. In the former case the maximum stresses may be easily determined, once the wind panel loads are known, by dividing the truss into two systems, as was done in the multiple system trusses previously considered.

In the latter case, the cross struts of the top system and the floor beams in the bottom system (in a through bridge) connect the two sets of diagonals so rigidly that it is impossible to divide into separate trusses; a reasonable assumption for such

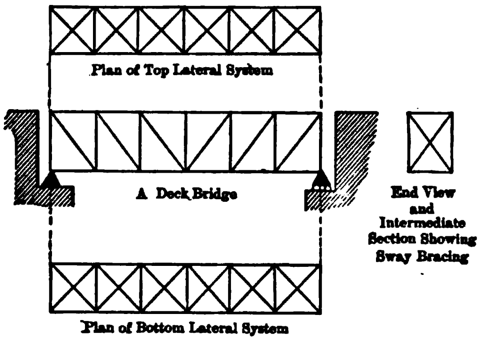


FIG. 181.

a truss is to consider the shear in a panel to be divided equally between the two diagonals, one being brought into tension and the other into compression.

It should be said that the present-day practice is to use riveted laterals in both top and bottom systems of railroad bridges in order to secure rigidity, but that tie rods are frequently used for highway bridges. Where the wooden floor bridge is continuous, as in many highway bridges, or where a continuous steel floor is used, the principal use of the lateral rods of the loaded chord system is to assist in erection by holding the trusses in line.

116. Approximate Determination of Maximum Stresses in Lateral Bracing.

Problem. Let the problem be the determination of the maximum stresses in the bottom lateral system of a through bridge with eight panels at 25 ft. and with trusses spaced 18 ft. between centres, assuming that the laterals are stiff members and able to carry both tension and compression. The horizontal lateral truss is shown in Fig. 182.

Solution. The lateral force acting at the bottom chord will be assumed as a moving force of 500 lbs. per lineal foot = 12,500 lbs. per panel. It is unnecessary to compute the lateral stresses in the floor beams, since the addition of a slight direct stress in these would be of no importance, hence it is immaterial whether this lateral force be assumed to be distributed between the two chords or be applied entirely to the windward chord. The latter condition will be assumed, however, for ease in computation. For convenience, the components of the diagonal stress at right angles to the axis of the truss will be spoken of hereafter as vertical components, and those along the truss axis as horizontal components.

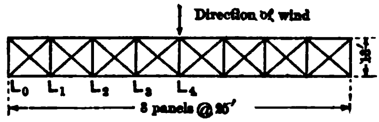


FIG. 182.

Index Stresses. These will be written for the full load, this being the simplest method of getting the chord stresses, and are shown in Fig. 183. The actual chord stresses will be $\frac{25}{18}$ of the index stresses.

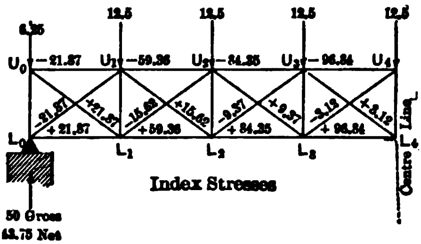


FIG. 183.

It should be noted that the lateral-truss chords are also the chords of the main truss, and that the wind stresses in them are sometimes of sufficient size to require additional area in these members, although it is customary to permit higher unit stresses for the combination of live, dead, and wind loads than would be allowable for live and dead stresses only.

Maximum Diagonal Stresses. The vertical components of the maximum diagonal stresses in 1000 lb. units will be as follows, assuming the shear in each panel to be divided equally between the two diagonals:

$$\text{Panel 0–1,}\quad \frac{1}{2}\left(3\tfrac{1}{2}\times 12.5\right)=+21.9;$$

$$\text{Panel 1–2,}\quad \frac{1}{2}\left(\frac{21}{8}\times 12.5\right)=+16.4;$$

$$\text{Panel 2–3,}\quad \frac{1}{2}\left(\frac{15}{8}\times 12.5\right)=+11.7;$$

$$\text{Panel 3–4,}\quad \frac{1}{2}\left(\frac{10}{8}\times 12.5\right)=+\ 7.8.$$

117. Portals. Approximate Solution. The portal bracing and end posts of a through bridge must be designed to carry to the abutment the reaction from the top lateral system, and also to withstand the wind pressure on the end posts themselves, the former being the more important factor. This combination of bracing and end posts is called the portal, and is a statically indeterminate structure. Accurate solutions of such structures may be made by the method of least work, but the approximate solution which follows is sufficiently accurate for all ordinary cases.

Fig. 184 shows a common type of end portal for a through bridge. The statical indetermination is due to the condition at the bottom of the end posts and to the rigidity of the portal bracing. Neither of the posts is pin-ended; that is, neither has a pin at

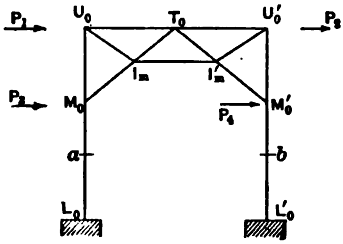


FIG. 184.

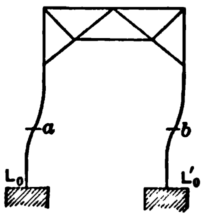


FIG. 185.

right angles to the plane of the portal, the main truss pins being in the plane of the portal. The ends of the posts are, however, really fixed to a considerable degree, since they bear upon the foundations, although they are not usually rigidly fixed thereto, and the dead weight of the structure is sufficient to offer a very considerable resistance to overturning under the action of the wind forces.

If the weight of the bridge is sufficient, it is evident that the posts may be treated as if they were fixed at the bottoms. Moreover, if the knee braces M_0I_m and $M'_0I'_m$ be rigidly fixed to the posts, the latter may be considered as fixed at points M_0 and M'_0 also. Assuming that such is the condition, the posts will bend under the action of the wind forces as shown in Fig. 185, and points of inflection will occur at a point in each post between the bottom of the knee brace and the bottom of the post. a and b indicate these points of inflection. If the position of these

points of inflection be known, and if the horizontal reaction at the bottom of the posts be also known, the stresses in the structure become determinate, since the moment at the point of inflection must equal zero.[1]

It is commonly assumed that each point of inflection occurs midway between the bottom of the knee brace and the bottom of the post. It is also commonly assumed that the portal bracing is so rigid that the distance apart of the posts remains unchanged under the action of the wind forces, and that in consequence the horizontal reaction at the bottom of each post equals one-half the sum of the applied loads. Neither of these assumptions is more than approximately correct, but in the ordinary structure the error introduced thereby into the design of the end posts is small, since the wind stresses in these members are in themselves small compared with the live and dead stresses, and the percentage error in consequence is still smaller. The portal bracing itself is frequently made considerably larger than is necessary, owing to the comparatively small magnitude of the wind forces, and the difficulty in choosing members with small enough areas which are also suitable in other ways.

With the points of inflection and the distribution of the horizontal reactions between the two posts known or assumed, the computation of the stresses in the various members may be easily made. The structure, however, differs somewhat from those which have been previously treated, since it consists of a combination of columns, carrying direct stresses and bending, and a truss.

The example which follows illustrates the method of computation based upon these assumptions.

Problem. Let the problem be the determination of the stresses in the portal of the bridge shown in Fig. 186.

---

[1] This may be proven as follows:

Let R = radius of curvature at any section of a member exposed to bending;
M = bending moment at this section due to external forces;
I = the moment of inertia at this section;
E = the modulus of elasticity.

From mechanics, $$\frac{1}{R}=\frac{M}{EI}.$$

At the point of inflection the beam must be straight, since at this point the curvature changes, hence R = infinity, and $\frac{1}{R}=0$, $\therefore$ $M=0$.

Solution. The wind force on top chord at, say, 200 lbs. per lineal foot of bridge equals 2500 lbs. per panel point per truss. The force applied by the lateral truss to the portal at m equals the vertical component in diagonal mo plus the panel load at m. The sum of these two forces equals $5000\times\frac{5}{2}+1250=13,750$ lbs. There will also be a force of 1250 lbs. at n. In addition to the wind force acting along the top chord, there will be

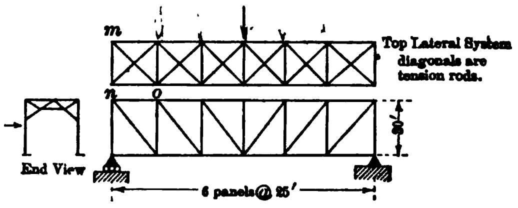


FIG. 186.

a uniformly distributed wind force applied directly to the end posts. This will be assumed as 100 lbs. per lineal foot of the member. The outer forces acting upon the portal will then be as shown in Fig. 187, assuming points of inflection and distribution of horizontal forces as previously stated. The vertical forces and bending moments at the bottoms of these posts may be computed as follows: Let the moment at the bottom of each post $=M$, and the vertical force $=V$. Then, since the moment at the point of inflection $=$ zero, the moment about an axis through the point of inflection in each post of the forces below that point

$$=10,500\times 11-100\times 11\times\frac{11}{2}-M.$$

$$\therefore\ M=109,450 \text{ ft.-lbs.}$$

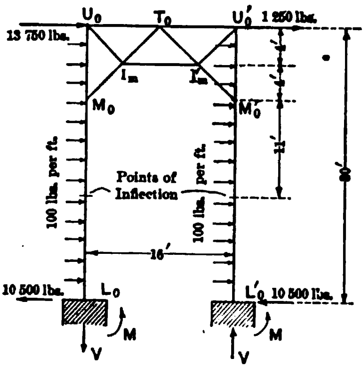


FIG. 187.

The direction of the moment in each case must be counter-clockwise, as shown, to balance the clockwise moment due to the horizontal forces.

In order that equilibrium may exist, the moment of the couple due to the vertical forces must equal the moment of the horizontal forces about any axis minus $2M$. Taking the origin of moments at the bottom of either post, the following equation may therefore be written:

$$15,000\times 30+6000\times 15-109,450\times 2-16V=0.$$

$$\therefore\quad V=20,070 \text{ lbs.}$$

The next step is the determination of the stresses in the portal members themselves, and the direct stresses, bending moments, and

shears in the end posts. It is evident that each main post is a continuous member without hinges. That is, the joint at M_0 can in no sense be considered a pin joint so far as the two sections U_0M_0 and M_0L_0 of this member are concerned, since the stability of the entire structure depends upon the lateral stability of these end posts. Indeed the moment in the post at this point, according to our hypothesis, equals $109{,}450-10{,}500\times22+100\times22\times11=-97{,}350$ ft.-lbs.[1] The other joints may, however, be pin joints, and will be so considered. Moreover the joint M_0 will also be considered a pin joint so far as the stress in M_0I_m is concerned; that is, the stress in M_0I_m will be assumed to be direct stress. To compute the stress in the portal bars it is necessary to treat the post U_0L_0 as a beam supported at the point M_0 by a truss bar, the direction of which determines the direction of the beam reaction at this point, and at the point U_0 by a reaction which is unknown in direction, and which equals the resultant of the unknown stresses in U_0T_0 and U_0I_m. This beam is loaded by a uniform load of 100 lbs. per foot over its entire length, and by the horizontal forces of 10,500 lbs. at L_0 and of 13,750 lbs. at U_0. It is also subjected at L_0 to a bending moment of 109,450 ft.-lbs., and a tension of 20,070 lbs.

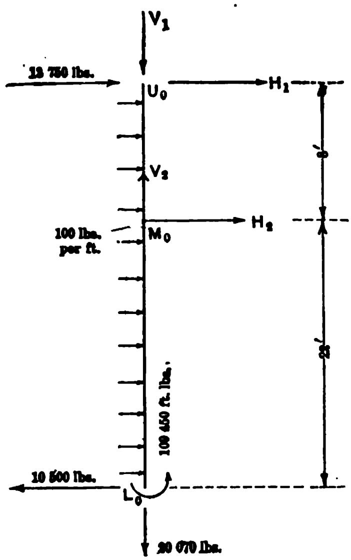


FIG. 188.

This condition is shown by Fig. 188, in which the reactions at U_0 and M_0 are represented by their horizontal and vertical components.

The ratio of V_2 to H_2 is determined by the slope of the portal bar M_0I_m. Since this makes an angle of 45° these two components are equal.

The ordinary equation of statics may now be applied. Application of the equation $\Sigma M=0$, using U_0 as the origin of moments, gives the following equation:

$$10{,}500\times30-3000\times15-109{,}450-8H_2=0;$$
$$\therefore\quad H_2=+20{,}070 \text{ lbs.}=V_2.$$

---

[1] This value may be verified by considering the portion of the post between M_0 and the point of inflection. The shear at the point of inflection in pounds $=10{,}500-1100=9400$, and the moment is zero, therefore, the moment in foot-pounds at $M_0=9400\times11-100\times11\times\frac{11}{2}=97{,}350$.

Application of the equation $\Sigma H=0$, gives the following equation:

$$-H_1-20{,}070-3000+10{,}500-13{,}750=0;$$
$$\therefore\ H_1=-26{,}320 \text{ lbs.}$$

Application of $\Sigma V=0$ gives the following equation:

$$V_1-20{,}070+20{,}070=0 \quad \therefore\ V_1=0.$$

Hence the stress in bar $U_0I_m=0$, from which it follows that the stress in I_mI_m'[1] $=0$, and that in $U_0T_0=-26{,}320$ lbs.

The actual stress in M_0I_m = stress in $I_mT_0=20{,}070\times1.414=+28{,}380$ lbs.

Proceeding in a similar manner with the other post, the following results are obtained:

Stress in T_0I_m' and $I_m'M_0'$ $=-28{,}380$ lbs.;
Stress in $I_m'U_0'=$ 0;
Stress in T_0U_0' $=+13{,}820$ lbs.

The computations may be checked by considering the joint T_0 and applying the equations of equilibrium. The forces acting at the joint are as shown in Fig. 189, and evidently satisfy the equations of equilibrium.

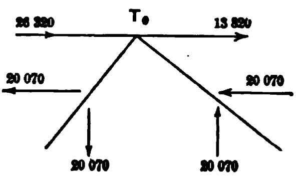


FIG. 189.

Since the stresses in U_0I_m, I_mI_m', and $I_m'U_0'$ are zero, it might perhaps be thought that these bars should be omitted, but it should be remembered that the computations are approximate, and that the stresses as determined by more exact methods may not be zero. Moreover the appearance of the portal is improved somewhat by the inclusion of these bars.

In addition to the determination of the stresses in the portal bars, the maximum moments and shears in the posts should also be obtained. These are alike for both posts and are shown by the curves of Fig. 190.

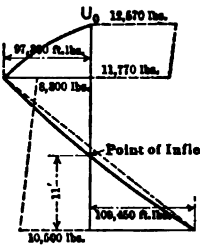


FIG. 190.—Curves of Moment and Shear in column. Full Line shows Moment.

The maximum direct stress in each post $=20{,}070$ lbs. It is tension in L_0M_0, compression in $L_0'M_0'$, and zero in M_0U_0, and $M_0'U_0'$.

Before leaving this subject, attention should be called to the fact that the wind forces cause stresses in the main truss members. These stresses are relatively small compared with the stresses due to the vertical loads, but may attain high absolute values in large trusses. In the windward trusses these stresses tend to cause compression in the bottom chord which in conjunction with the stresses due to longitudinal thrust caused

[1] This value may be checked by taking moments about T_0 of the forces to the left of a vertical section through this point. It will be found that this moment $=0$, hence the stress in I_mI_m' equals zero.

by the tractive force may even reverse the normal tension in the end bottom chord members, which are frequently made as columns to resist this compression.

118. Portals—Miscellaneous. The portal treated in Art. 117 represents a common type of portal which is statically determined with respect to the inner forces. Portals are frequently built, however, which are statically undetermined with respect to the inner as well as the outer forces. For such cases the methods used in the treatment of double-system trusses may sometimes be applied. For more complicated portals special methods may have to be devised, but the construction of such portals should be avoided.

Portals which lie in a plane inclined to the vertical, as would be the case for a bridge with inclined end posts, may be treated in the same manner as vertical portals, care being taken to use the correct lengths along the posts and not the vertical projections of these lengths.

119. Transverse Bents in Mill Buildings—Approximate Method. A typical structure of this type is illustrated by Fig. 191. The stresses due to the vertical forces may be figured in the ordinary manner, assuming vertical reactions at points b and i, and zero stress in knee braces ac and hk. To determine the horizontal wind forces an approximate method similar to that used in computing portals is commonly employed, it being assumed that the horizontal reactions at the bottoms of the columns are equal and that points of inflexion occur midway between bottoms of columns and points of connection between columns and knee braces.[1] As in the portals all joints are assumed to be pin-joints except those at a and k, while these

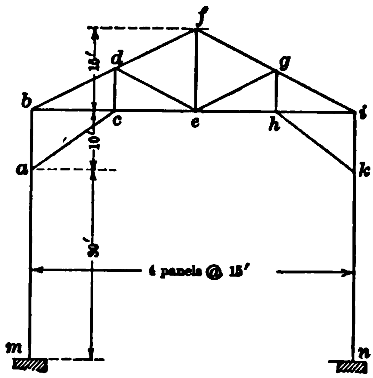


FIG. 191.

[1] This assumption should not be made unless warranted by the conditions existing at the bases of the columns. In many structures of this character the resistance to bending-moments offered by the column footing is very slight; in such cases the point of inflexion may be assumed as occurring at the base of the column.

latter are also so considered with respect to the stresses in the knee braces themselves, which are assumed to act along the axes of these bars.

The stresses in bars *ac* and *kh* may be determined as in the portal by applying the equation, $\Sigma M=0$, to the two columns *bm* and *in*, using for the origin of moments points *b* and *i*. The horizontal and vertical forces required at points *b* and *i* may then be obtained by the application of the equations, $\Sigma V=0$ and $\Sigma H=0$, to the two columns. With these values determined, the roof truss may be treated as any simple truss, the outer forces being the applied wind loads, the stresses in the knee braces, and the reactions at the column tops.

The complete determination of the stresses in such a structure by the approximate method will not be given, the problem which follows including all the essential points. An accurate determination of these stresses may be made by the theorem of least work, but will not be given here.

Problem. Compute the horizontal and vertical components of the truss reactions at points *b* and *i*, and of the knee-brace stresses in the transverse bent, shown in Fig. 191, for a horizontal wind force of 600 lbs. per lineal foot on *bm*, and a normal wind force of 400 lbs. per lineal foot on *bf*.

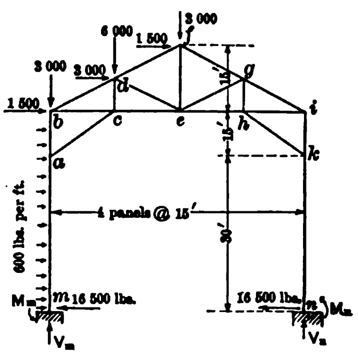


FIG. 192.

Solution. The applied loads will be as shown in Fig. 192. As in the portal the horizontal components at *m* and *n* are each assumed to equal one-half the total horizontal force on the structure, thus having a value of 16,500 lbs. each. For this case the moment at point *m* will not equal that at point *n*, since no wind force is assumed to act on the leeward column. Each moment may be found, however, by applying the equation, $\Sigma M=0$, about the point of inflection of the forces below that point. The equations for these moments are as follows:

$$16{,}500\times 15-600\times 15\times \frac{15}{2}-M_m=0,$$

From which $M_m=+180{,}000$ ft.-lbs.

In a similar manner $M_n=16{,}500\times 15=247{,}500$ ft.-lbs.

The vertical reaction V_m may now be determined by the application of the equation, $\Sigma M=0$, using for an origin the point n. The resulting equation is as follows:

$$-180{,}000+60V_m+600\times45\times22.5+6000\times52.5-12{,}000\times45-247{,}500=0.$$

From which $V_m=+750$ lbs.

Application of $\Sigma V=0$ gives $V_n=12{,}000-750=+11{,}250$ lbs.

The horizontal components in bars ac and hk may next be computed by applying the equation, $\Sigma M=0$, using points b and i respectively as origins of moments.

The equations thus obtained are as follows:

$$180{,}000+600\times45\times\frac{45}{2}+HC\text{ (bar }ac)\times15-16{,}500\times45=0$$

and

$$247{,}500-HC\text{ (bar }hk)\times15-16{,}500\times45=0$$

From which HC (bar ac) $= -\ \ 3{,}000$ lbs.

and HC (bar hk) $= -33{,}000$ lbs.

The vertical components of these forces equal the horizontal components, since the bars have a slope of 45°.

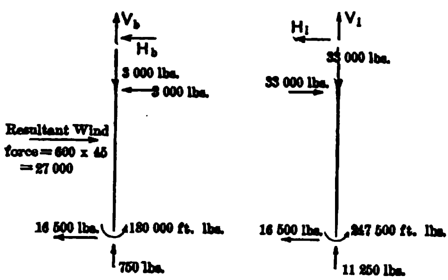


FIG. 193.

The reactions at points b and i may now be determined by applying the equations $\Sigma H=0$ and $\Sigma V=0$ to each column as a whole. The forces acting on the columns are shown in Fig. 193, hence,

$$H_b=27{,}000-19{,}500=+7500 \text{ lbs.}$$

$$V_b=+\ \ 2{,}250 \text{ lbs.}$$

$$H_i=+16{,}500 \text{ lbs.}$$

$$V_i=+21{,}750 \text{ lbs.}$$

The forces acting on the truss will therefore be as shown in Fig. 194, and the truss may now be computed in the ordinary manner. The

moments, shears, and direct stresses in the columns may be determined as in the portal columns previously treated.

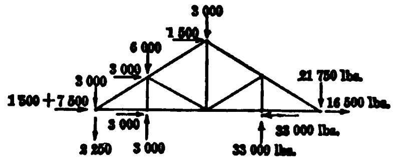


FIG. 194.

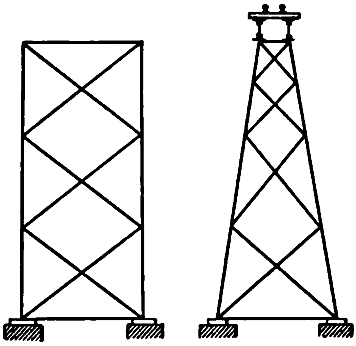
120. Viaduct Towers. In the determination of the stresses in such towers it is necessary to consider the vertical forces due to the weight of the structure itself and the superimposed load, and the horizontal forces due to wind, centrifugal force if the structure be curved, and tractive force. Such towers are usually composed of four columns, braced transversely and longitudinally. two of the columns being supported on planed base plates and being free to move horizontally in the direction of the longitudinal axis of the structure. To obtain sufficient width at the base to prevent excessive uplift at windward columns when the structure is either unloaded, or carrying an empty train, the latter are usually built to a batter, transversely, of one horizontal to six vertical. For symmetry a double system of bracing should be used, and the structure will, therefore, be statically undetermined unless the bracing consists of rods, which is not common in railroad viaducts. The stresses due to the horizontal forces may be computed in a manner similar to that used for the wind bracing systems already computed, i.e. by assuming each diagonal to carry one-half the stress it would be called upon to carry if but one system of diagonals were used. If the vertical loads are symmetrical with respect to the central axis of the tower they will cause no primary stress in the bracing, and the vertical components in the columns will therefore equal the vertical loads. If the vertical loads are not symmetrical with respect to the tower, as may be the case with a structure built on a curve, stresses due to these loads will be caused in the diagonals as well as the columns.

The necessary computations for the loaded structure are clearly illustrated by the problem which follows.

Problem. Compute stresses for the tower shown in Fig. 195 due to the following assumed loads:

Dead load, 600 lbs. per foot per rail for girder and track.
200 lbs. per foot in height of tower for each column.
Live load, 3000 lbs. per foot per rail.

Wind load, 600 lbs. per lineal foot of structure applied at base of rail and 50 lbs. per foot of height of tower for each column.
Tractive force, 20% of live load applied at base of rail.



Viaduct Tower

FIG. 195.

The spans on each side of tower are each 56 ft. in length centre to centre of end bearings.

Dimensions and load concentrations are shown in Fig. 196.

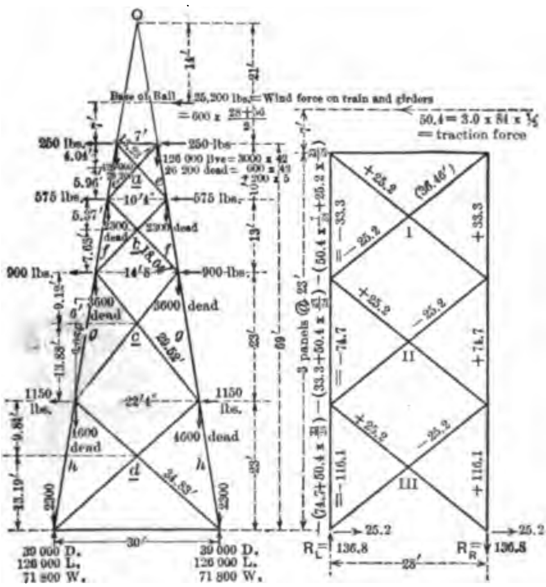


FIG. 196.

And the necessary computations follow:

COMPUTATION OF STRESSES IN VIADUCT TOWER

TRANSVERSE BRACING

Bars.	Horizontal Component.	Stress.
a	$\frac{25{,}200\times14+500\times21}{2\times25.04}=\frac{363{,}300}{50.08}=7250$	$7250\times\frac{13.23}{8.67}=\pm11{,}000$
b	$\frac{363{,}300+1150\times31}{2\times36.37}=\frac{399{,}000}{72.74}=5500$	$5500\times\frac{18.04}{12.50}=\pm\ 7900$
c	$\frac{399{,}000+1800\times44}{2\times53.12}=\frac{478{,}200}{106.24}=4500$	$4500\times\frac{29.52}{18.50}=\pm\ 7200$
d	$\frac{478{,}200+2300\times67}{2\times76.81}=\frac{632{,}300}{153.62}=4100$	$4100\times\frac{34.83}{26.17}=\pm\ 5500$

The column stresses are shown on next page.

The maximum uplift on the windward column should also be determined. For the wind load previously considered the uplift at base of column $=71{,}800$ lbs. To this should be added the uplift due to the tractive force. The uplift of a loaded train due to tractive force $=\frac{3000}{2}\times84\times0.20\times\frac{76}{28}=68{,}400$ lbs. The total reaction on one column due to live and dead loads

$$=39{,}000+\frac{3000}{2}\times42=102{,}000\text{ lbs.},$$

hence the net uplift $=(71{,}800+68{,}400)-102{,}000=38{,}200$ lbs. It is also common to determine the uplift on the unloaded structure due to an assumed wind force of 50 lbs. per sq. ft. on one and one-half times the vertical projection of the structure.

In the design of viaduct towers it is common to assume that the combination of dead, live, wind, and tractive forces will seldom, if ever, occur simultaneously, and that in consequence a higher unit stress may be used for these combined forces than would be employed for live and dead stresses only, a common practice being to increase the unit stress 25%. For example, if the allowable unit stress for dead load, and live load corrected for impact, is 16,000 lbs., a value of 20,000 lbs. would be used for the maximum stresses due to live, dead, wind, and traction. If centrifugal force exists the stresses due to it should be considered as live stresses, but need not be corrected for impact.

COMPUTATION OF STRESSES IN VIADUCT TOWER

COLUMNS

Bars.	Dead Stress.	Live Stress.	Wind Stress.		
			Moments about Intersection of Diagonals in each Panel.	Vertical Components of Stress.	Stress.
e	$26{,}200\times1.014$ $=-26{,}600$	$126{,}000\times1.014$ $=-127{,}700$	$25{,}200\times11.04+500\times4.04$ $=280{,}200$	$280{,}200\div\left[7+\frac{1}{3}(4.04)\right]$ $=33{,}600$	$33{,}600\times1.014$ $=\pm34{,}000$
f	$28{,}500\times1.014$ $=-28{,}900$	$-127{,}700$	$280{,}200+25{,}700\times11.33+1150\times5.37$ $=577{,}600$	$577{,}600\div\left[10.33+\frac{1}{3}(5.37)\right]$ $=47{,}600$	$47{,}600\times1.014$ $=\pm48{,}200$
g	$32{,}100\times1.014$ $=-32{,}600$	$-127{,}700$	$577{,}600+26{,}850\times16.75+1800\times9.12$ $=1{,}043{,}800$	$1{,}043{,}800\div\left[14.67+\frac{1}{3}(9.12)\right]$ $=58{,}900$	$58{,}900\times1.014$ $=\pm59{,}700$
h	$36{,}700\times1.014$ $=-37{,}200$	$-127{,}700$	$1{,}043{,}800+28{,}650\times23.69+2300\times9.81$ $=1{,}745{,}000$	$1{,}745{,}000\div\left[22.33+\frac{1}{3}(9.81)\right]$ $=68{,}200$	$68{,}200\times1.014$ $=\pm69{,}200$

LONGITUDINAL BRACING

The components of the stress in the diagonals are shown in Fig. 196 and were obtained by the method of shear. The actual stress may be obtained from these components in the usual manner and will not be given here.

If the vertical loads are not applied to the tower symmetrically they will also cause stresses in the diagonal bracing, since their moment about the point O will no longer equal zero. Such a condition will usually occur if the viaduct be located on a curve, in which case the eccentricity will be due not only to the eccentricity of the centre line of the track, but also to the shifting laterally of the centre of gravity of the train by the superelevation of the outer rail. The computations for such a case present no difficulty and will not be given.

If rods are used for the diagonal bracing it will be necessary to use horizontal struts between panel points and only one system of rods will be in action at one time. The computations will be simplified somewhat for this case, but the same general mode of procedure may be adopted. It is frequently assumed in designing such towers that, even when rigid bracing is used, but one diagonal will be in action at any one time in a panel.

The methods given in this article are approximate, but are sufficiently accurate for ordinary cases.

PROBLEMS

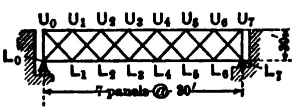
54. *a.* Write the index stresses for this truss for the following dead loads:

Top chord .... 1200 lbs. per foot
Bottom chord .... 600 "

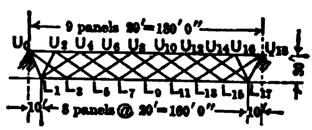
b. Compute maximum live stress in bar U_1L_2 for a uniform live load of 3000 lbs. per foot and a locomotive excess of 30,000 lbs., all on top chord.

c. Draw influence line for stress in bar U_2U_3 and compute from it the maximum live stress for the loads given in *b.*

d. Compute maximum live stress in bar U_2U_3 for loads given in *b* by the use of index stresses.



PROB. 54.



PROB. 55.

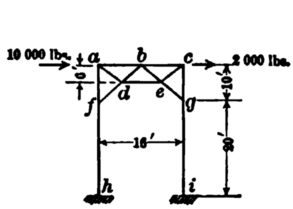
55. *a.* Make a sketch of the truss, showing by different colors the systems into which you would divide it.

b. Write the index stresses, using the dead loads of previous problem.

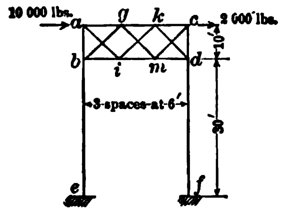
c. Draw an influence line for the stress in bar L_7L_9.

56. *a.* Compute the stress in all portal bars, and the maximum moments, shears, and direct stresses in the columns of this portal for the applied forces shown, assuming that the moment at points h and i = zero, and that diagonals can carry both tension and compression.

b. Compute the stresses in the same bars, assuming that the point of inflection in each post is one-quarter of the distance up from the bottom of the column to the bottom of the knee brace.



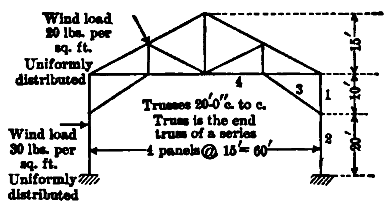
Prob. 56.



Prob. 57.

57. Determine stresses in all bars of this portal, and maximum moments, shears, and direct stresses in the columns, assuming points of inflection in each column at a point midway between the bottom of the column and the bottom of the portal.

58. Assuming the columns in this transverse bent to have points of inflection at bottom, compute maximum bending moment in right-hand



Prob. 58.

column and maximum shear in sections 1 and 2. Compute also maximum stresses in bars 3 and 4 and state their character.

CHAPTER IX.

CANTILEVER BRIDGES.

121. Types of Structures for Long-span Bridges. Where the expense of constructing foundations or the restrictions of navigation prohibit the use of spans shorter than 600 to 700 ft., other types of structures than the simple end-supported spans which have been previously considered are more economical and are commonly used. Three types of such structures are frequently employed, viz., the steel arch, the suspension bridge, and the cantilever bridge. Of these the suspension bridge is highly indeterminate and will not be considered at this point. The arch and cantilever may be either determinate or indeterminate, but only the former types will be treated in this chapter.

122. Cantilever Bridges Described. In the construction of cantilever bridges, advantage is taken of the fact that a span with one or two projecting arms may be erected by constructing false work under the main span only, the projecting ends being gradually built out from the supporting piers, their weight being balanced by the weight of the main span. A long bridge of this character may therefore be erected by the use of falsework under every other opening only. For example, the bridge shown in Fig. 195 may be built by using falsework from m to n and from o to q, the channel span no being sustained during erection by the weight of the anchor arms mn and oq and by anchorage at m and q. A number of large bridges of this type have been constructed in the United States, among which may be mentioned the Red Rock Bridge, the Poughkeepsie Bridge, and the Beaver Bridge, the outline of which is shown in Fig. 195. The Harvard Bridge at Boston is a plate-girder cantilever. The Forth Bridge in Scotland, with a clear span of 1710 feet, the longest clear span in the world, is also a cantilever bridge. If the

suspended span be omitted and the cantilever arms connected, the bridge becomes indeterminate. The Queensboro Bridge in New York city is the most important example of a bridge of

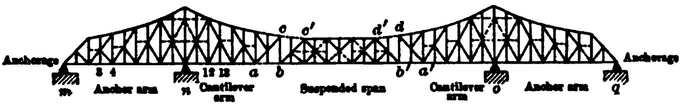


FIG. 195.

this type. Such a bridge can be built along more graceful lines than if a suspended span is used, and troublesome details at the connection of the suspended span and the cantilever arm avoided.

123. Equations of Condition. Let Fig. 196 represent, diagrammatically, a bridge of three spans similar to that shown in Fig. 195. As shown, there are eight unknown components of the outer forces, viz., two at each point of support. Evidently this structure is statically indeterminate to a high degree. The most obvious method of reducing the degree of indetermination is to fix the direction of some of the reactions. Since one of the

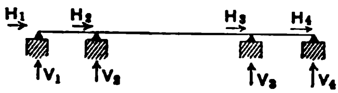


FIG. 196.

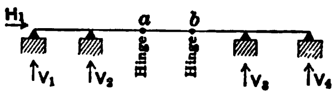


FIG. 197.

reactions, at least, must have a horizontal as well as a vertical component to give stability, it is possible in this manner to eliminate only three of the eight unknowns, this being insufficient to make the structure statically determined. The remaining equations necessary to secure statical determination must be obtained by the method of construction and are called equations of conditions. Such equations may be obtained by introducing hinges in the end spans, as indicated in Fig. 197, these hinges being so constructed as to make it impossible to transmit bending moment through them. This construction therefore gives the two following additional equations:

$$\Sigma M_a = 0 \quad \text{and} \quad \Sigma M_b = 0.$$

These equations signify that the moment of all the forces on *either* side of either hinge about an axis passing through the hinge at right angles to the plane of the forces equals zero. These

equations should not be cônfounded with the general equation $\Sigma M=0$, which is also applicable at the hinges, but which means that the moment of all the forces on *both* sides of any section about *any* axis perpendicular to the plane of the forces equals zero.

With regard to the moment about the hinges it should be noted that although the moment about a of all the forces to the left or right thereof $=0$, it should not be supposed that this gives two independent equations, sinee if the moment about a of all the forces to the left of a equals zero, the moment of all the forces to the right of a about the same point must also equal zero, such a result following at once by the subtraction of the former equation from the general equation $\Sigma M=0$. There are then but five entirely independent equations, hence five and only five unknown quantities can be determined, and the structure is, therefore, determinate. Were there more than five independent equations the structure would be unstable; were there less it would be statically undetermined.

The simplest method of providing a hinge in a truss is to omit a chord bar in one panel as is done in Fig. 198. Evidently the

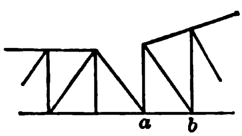


FIG. 198.

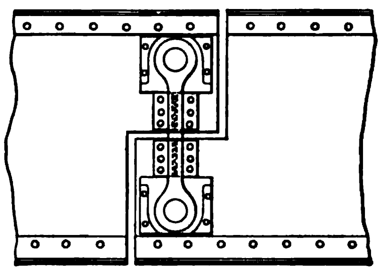


FIG. 199.

moment about an axis through a of all the forces on either side of a must equal zero, since the structure can by its construction offer no resistance to moment at this point.

For a plate girder cantilever a hinge may be constructed as shown in Fig. 199.

As it is uneconomical in practice to have a long cantilever bridge restrained horizontally at one point only,—such a condition involving the transmission of a horizontal force, if applied at one end of the structure, throughout the entire length of the bridge—it is common to fix the structure horizontally at more than one

pier and to omit a bottom chord bar at one or more of the hinges, thereby preventing the transmission of horizontal forces across the bridge. For the case illustrated by Fig. 198 the omission of bar *ab* would accomplish this result.

The application of these methods to the structure shown in Fig. 195 would involve the omission of bars *cc'*, *d'd*, and *ab*, and of the rollers at *o*. In practice the bars mentioned would not actually be omitted, as they are necessary in erection and improve the appearance of the structure. They should, however, be made adjustable and incapable of resisting a horizontal thrust.

124. Anchorage. Since the load on the suspended span or on the cantilever arms causes negative reactions at points such as *m* and *q*, Fig. 195, which may exceed the dead reactions at these points, it is necessary to anchor the structure to the masonry and to provide sufficient weight in the piers to equal this uplift. The anchorage usually consists of girders embedded in the pier and fastened to the structure by eye-bars. The freedom to move horizontally may be obtained by rollers or other devices.

125. Reactions—Cantilever Trusses. The reactions upon structures of this type may be determined by the application of the three equations of statics combined with the equations of condition in the same general manner as for simple trusses and girders. The problem is, however, more complicated than for structures supported at two points, and in consequence influence lines for certain of the reactions in typical cantilevers will be given and methods of determining the reaction values stated.

Consider first the structure illustrated by Fig. 195. For this cantilever the trusses *mbc* and *db'q* evidently act like beams supported at two points only and supporting the suspended span, *bb'* at their ends. That this follows from the application of the equations of equilibrium and condition may be proven in the following manner:

Assume first a concentrated load, P, upon the truss *db'q*. For this condition the forces acting to the left of b are the same as the forces acting to the left of b', viz., V_1 and V_2, Fig. 200, and the moment of these forces about each of the hinges b and $b'=0$.

$\therefore$ the following equations may be written:

$$\text{I.}\quad V_1(L_1+L_2)+V_2L_2=0.$$

$$\text{II.}\quad V_1(L_1+L_2+L_3)+V_2(L_2+L_3)=0.$$

Subtracting I from II gives $V_1L_3+V_2L_3=0$.

$\therefore\quad V_1+V_2=0,\quad$ hence $\quad (V_1+V_2)L_2=0$.

Subtracting this latter equation from I gives $V_1L_1=0$, therefore, $V_1=0$, $V_2=0$, and $V_1'+V_2'=P$, hence the span $db'q$ when loaded, acts like a simple beam, since the moment at each end is zero, and the sum of the reactions equals the load.

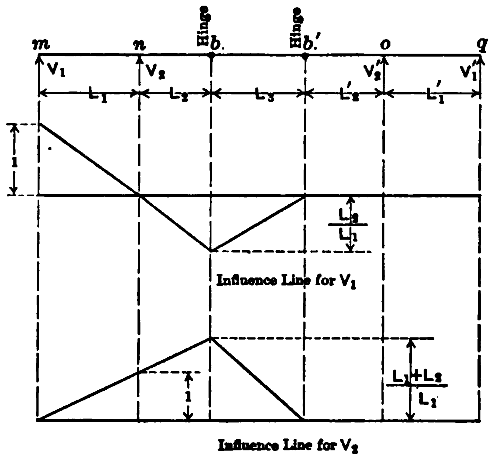


FIG. 200.

Now consider a load, P, on span bb' at a distance x from b'. The following equations may be written for the moments about b and b' respectively.

$$\text{III.}\quad V_1(L_1+L_2)+V_2L_2=0.$$

$$\text{IV.}\quad V_1(L_1+L_2+L_3)+V_2(L_2+L_3)-Px=0.$$

Subtracting III from IV gives $V_1+V_2=\dfrac{Px}{L_3}=$positive shear at b. The moment at n equals the moment at b plus the product of the shear at b and the lever arm L_2; it therefore equals $-\left(\dfrac{Px}{L_3}\right)L_2$, the moment at b being zero by construction. This moment also equals $-V_1L_1$, hence the following equation may be written:

$$\left(\frac{Px}{L_3}\right)L_2+V_1L_1=0,$$

whence
$$V_1=-\left(\frac{Px}{L_3}\right)\left(\frac{L_2}{L_1}\right).$$

This is identical with the reaction that would be obtained if the span bb' were assumed to be supported on the ends of the two simple beams mb and qb'. In a similar manner the reaction at q may be shown to equal the reaction that would exist if a similar assumption were to be made; hence the proof of the statement that the reaction in a structure such as that shown is identical with the reactions which would occur if the structure were to be considered as composed of two independent beams mb and qb', supporting the simple span bb' at their ends.

The influence lines in Fig. 200 show clearly the reactions due to loads in the different portions of the structure. It should be noted that these influence lines would be unchanged and statical determination accomplished by omitting the rollers at o and bar ab in Fig. 195.

The cantilever shown in Fig. 201 differs somewhat from that of Fig. 195, some of the equations of condition for the structure

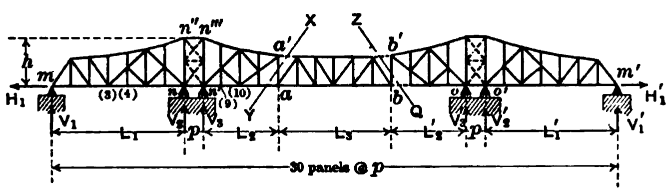


FIG. 201.

being established by the omission of diagonals over the central piers. The structure as shown has eight unknown reactions; six vertical and two horizontal. To determine these unknowns there are available in addition to the three general equations of statics, five equations of condition. Two of these condition equations are obtained by the insertion of hinges at a and b, i.e., by the omission of upper chord bars at these points; two by the omission of diagonals over the centre piers,[1] and one by the omission of the bottom chord bar in the cantilever arm adjoining the hinge at b. Of these condition equations the latter is of use in determining reactions due to horizontal forces only (wind or

[1] Diagonals may be used in the towers for purposes of bracing, but they should be of small size and offer slight resistance to distortion. This same device is frequently employed in partially continuous draw spans.

longitudinal thrust of train), and in combination with the equation $\Sigma H = 0$ is sufficient for this purpose. The remaining four equations of condition and two of statics may be used in the following manner to determine the six vertical reactions.

Let S = shear in panel $nn' = 0$ by construction.
M_n = moment at n.
M_n' = moment at n'.
M_a = moment at $a = 0$ by construction.
M_b = moment at $b = 0$ by construction.
S_a = tension in hanger aa'.

Case 1. Load on suspended span ab. Consider the load P at distance x from b, and the portion of the structure between sections XY and ZQ. The following equation may be written:

$$M_b = Px - S_a(L_3) = 0,$$

$$\therefore \qquad S_a = \frac{Px}{L_3}.$$

But S_a is the supporting force at the left end of the suspended span and equals the reaction at the corresponding end of a simple end-supported span. It is evident, therefore, that the suspended span ab acts like a simple end-supported truss, since the moment at each end equals zero and the reactions are inversely proportional to the distance of the load from either end. It should be observed that no stress is caused in the hangers at a and b by a load unless it is applied to the suspended span.

Case 2. Load P on cantilever arm $n'a$ at distance x from n'. For this case

$$M_a = M_n' + (S + V_3)L_2 - P(L_2 - x),$$

$$M_n' = M_n + Sp,$$

$$V_1 + V_2 = S,$$

$$M_n = V_1 L_1,$$

$$S_a = 0.$$

By construction $M_a = 0$, and $S = 0$.

$$\therefore \qquad M_n' = M_n \text{ and } V_1 = -V_2,$$

and

$$M_n' + V_3 L_2 - P(L_2 - x) = 0,$$

hence, $$V_1L_1+(V_3-P)L_2+Px=0.$$

But $$V_1+V_2=S=0,$$

and

$$V_3-P=S, \quad \text{since} \quad S_a=0; \quad \text{hence} \quad V_3-P=0 \quad \text{and} \quad V_3=P$$

$$\therefore \qquad V_1L_1=-Px,$$

Hence $$V_1=-\frac{Px}{L_1}=-V_2.$$

It follows that for a load on the cantilever arm, an', V_1 equals the reaction that would occur on the truss ma if points n and n' coincided; $V_2=-V_1$, and $V_3=P$.

Case 3. Load P on anchor arm mn at distance x from n.

For this case

$$Sa=0,$$

$$M_n=M_n'=0,$$

$$V_3=0.$$

Also, $$V_1L_1-Px=M_n=0,$$

and $$V_1=\frac{Px}{L_1}.$$

Hence for this case the anchor arm mn acts like a simple span supported at m and n.

Fig. 202 shows influence lines for reactions V_1, V_2 and V_3 and should offer no difficulty to the student.

From the influence lines of Fig. 202 it is evident that for a uniform live load in a cantilever like that shown in Fig. 201 the maximum upward reaction at m occurs with mn fully loaded, and maximum negative reaction with $n'b$ fully loaded. Also that V_2 is always upward and has its maximum value when the load extends from m to b, while V_3 is also upward for any loading and has a maximum value with load from n to b.

126. Shears and Moments—Cantilever Trusses. With the reactions known the shears and moments at any section of a cantilever truss may be readily determined. The influence lines in Figs. 203 and 204 show clearly the variations in these functions

for certain typical portions of the anchor and cantilever arms of the trusses shown in Figs. 195 and 201. No influence lines are drawn for the suspended spans since these, as has been shown, may be treated like any simple span.

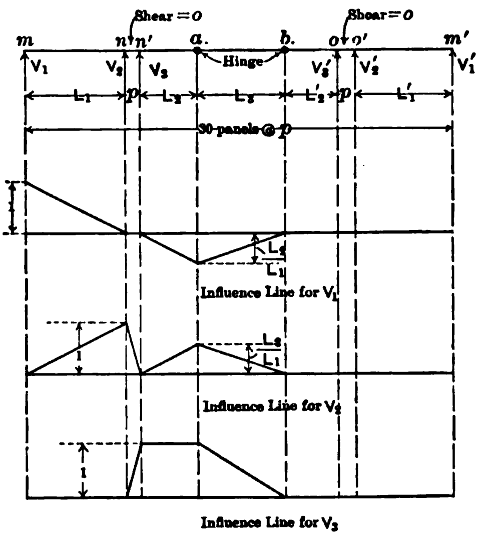


FIG. 202.

127. Bar Stresses—Cantilever Trusses. The determination of the bar stresses in cantilever trusses involves no special difficulties and may be accomplished by the use of the methods employed for simple trusses. Influence lines may be employed for determining the position of the loads, if concentrated load systems are to be used. For structures of the magnitude and weight of such bridges, however, the actual use of concentrated load systems for the stresses in the main truss members is generally unnecessary, an equivalent uniform load giving nearly if not quite as accurate results.

The determination of the position of a uniform live load for maximum stress in each bar and the computation of that stress may be accomplished by the use of influence lines if desired. An influence table similar to that prepared for the three-hinged arch given later showing the stress in each bar for a load at every panel point should, however, generally be prepared to facilitate

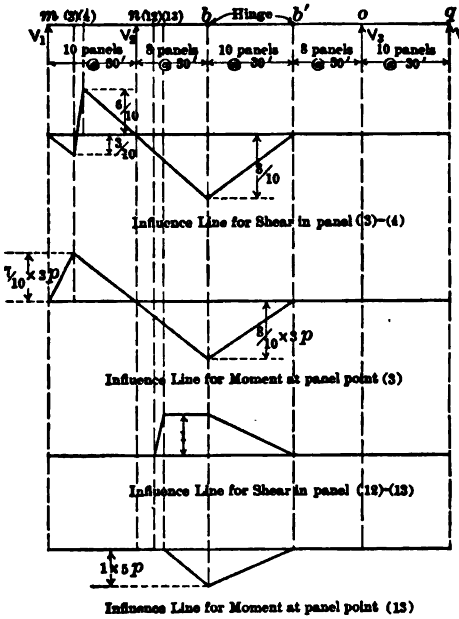


FIG. 203. (Refers to truss on page 259.)

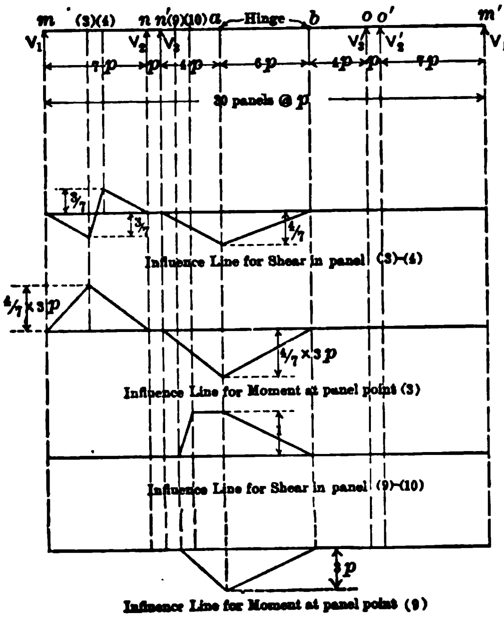


FIG. 204. (Refers to truss on page 263.)

the computation of the stress due to the dead load, which in a large structure should not be taken as uniformly distributed, and with this table once prepared no advantage would be gained by using influence lines. The influence line in Fig. 205 is given to illustrate the variation in stress in a particular bar rather than for its aid in computing the stress. This statement also applies to the influence lines of the previous articles.

If it be desired to use influence lines to check the tabular results the actual stress may be determined most readily for uniform loads by multiplying the areas between the influence line and the horizontal axis by the proper load.

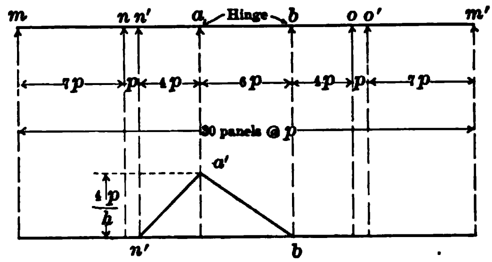


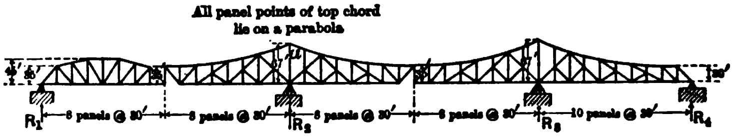
FIG. 205.—Influence Line for Stress in Bar $n''\ n'''$, Truss shown in Fig. 201.

Referring to Fig. 205, it is evident that the maximum stress in bar $n''\ n'''$ will occur with the truss loaded with the uniform live load from n' to b, and that its value equals the product of the area $n'a'b$ and the combined live and dead loads per foot, provided these are uniformly distributed.

PROBLEM

59. *a*. Show that this structure is statically determined with respect to the outer forces.

b. Draw influence line for reaction at R_2.



PROB. 59.

c. Draw influence line for stress in bar *a* and compute its maximum value for a uniform live load of 3000 lbs. per ft.

CHAPTER X.

THREE-HINGED ARCHES.

128. Characteristics of the Arch. The essential difference between the ordinary arch and the girders and trusses that have hitherto been investigated is that the stresses in an arch may be confined to compression and shear, while in trusses and girders large tensile stresses are also developed. This possible elimination of tensile stress in the ordinary arch rib is due to the fact that both ends of the arch are fixed in position by construction, hence each reaction has a horizontal component even under vertical loads; in consequence the reactions converge, and if the shape and thickness of the arch rib be properly chosen, the resultant force at each section for any given position of the loads may be made to pass through the centre of gravity of the section and therefore cause no bending moment, or so near the centre of gravity that the tensile fibre stress due to the bending moment caused by the eccentricity is insufficient to overcome the compression due to the thrust.

The advantage of the arch form was well known to the ancients, as is shown by the many stone arches constructed by the Romans and even by older races, and the arch remains to the present day one of the most useful and graceful of structures, its employment being frequently dictated both by æsthetic and utilitarian considerations.

129. Types of Arches. Up to a comparatively recent period the arch was always constructed as a statically undetermined structure, similar to that shown in Fig. 206, which represents the conventional masonry arch with neither reaction fixed in direction, magnitude, or point of application, the arch being in consequence statically undetermined in a three-fold degree, having six unknowns.

With the application of iron and steel to bridge construction came a recognition of the advantage of statical determination,

and metal arches began to be constructed in which some of the unknowns were eliminated by the insertion of hinges. Such

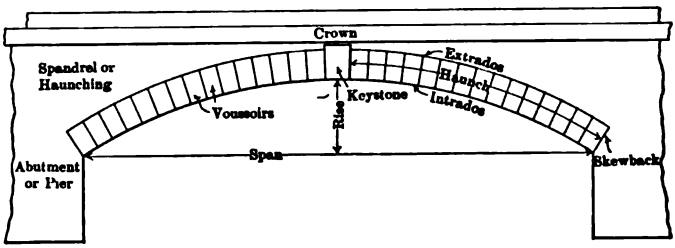


FIG. 206.—Masonry Arch.

arches are shown in Figs. 207 and 208. If in the arch shown in Fig. 208 a hinge be inserted at the centre similar to that of the arch shown in Fig. 207, the arch becomes a three-hinged arch and

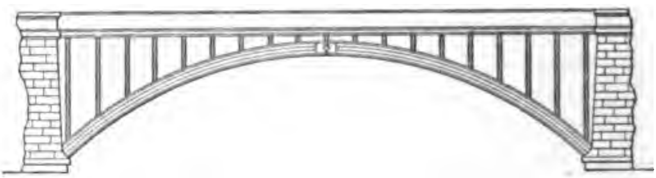


FIG. 207.—Metal Arch with One Hinge.

is statically determined. The ribs of metal arches may be formed either of plates and angles as in plate girders; of cast iron or cast steel segments riveted together; or of riveted trusses.

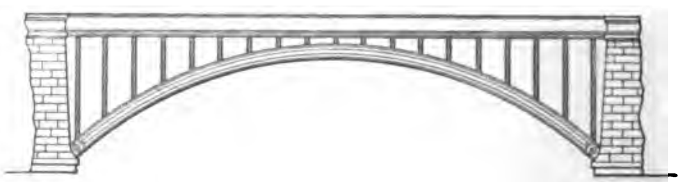


FIG. 208.—Two-hinged Metal Arch.

In recent years a considerable number of three-hinged masonry arches have been constructed, and since the adoption of plain

and reinforced concrete to arch design, the custom of applying the loads at fixed points to the arch rib by transverse walls has also been adopted in many long-span bridges, thus doing away with some of the uncertainty which formerly occurred in such cases, and securing many of the advantages of the metal arch. Fig. 209 illustrates such an arch. It should be said, however, that the common type of masonry arch is that without hinges, shown by Fig. 206.

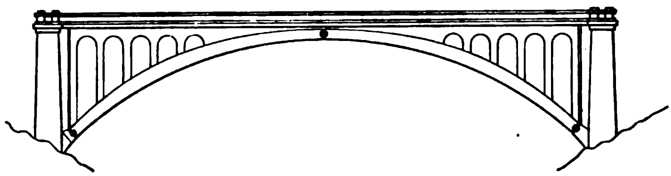


FIG. 209.—Three-hinged Masonry Arch.

One other important type of metal arch—the spandrel-braced arch, is shown by Fig. 210. This structure is in reality a combination of a truss and an arch rib. As will be shown later, if the arch rib in the three-hinged spandrel-braced arch be constructed to a parabolic curve, the diagonals and top chord will not be in action under a full uniform load, the arch rib in that case acting like the arches previously described, the loads being applied through the vertical posts.

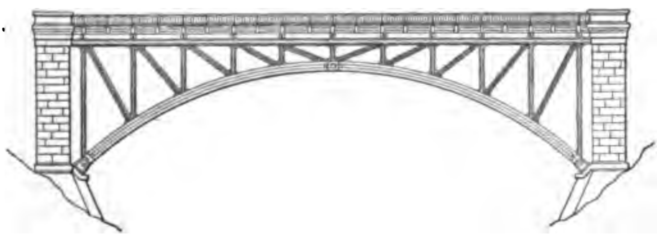


FIG. 210.—Three-hinged Spandrel-braced Metal Arch.

Like the other arches the spandrel-braced type is frequently constructed as a two-hinged arch. The three-hinged arch is the only type which will be considered here, the statical indetermination of the other forms requiring the development of other than statical methods as a preliminary to their investigation.

130. Reactions—Three-hinged Metal Arches. These may be computed for any position of the load by the application of the three general equations of statics combined with the equations of condition established by the hinges.

If the end supports are at the same elevation, as is generally the case, the horizontal components of the reactions balance

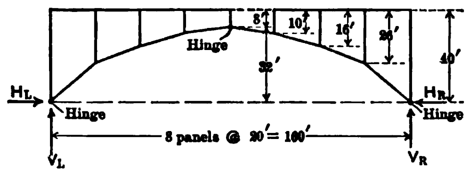
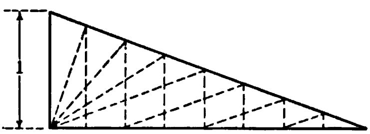


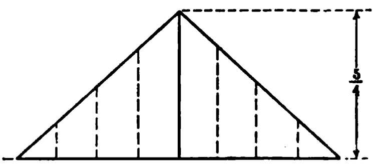
FIG. 211.

and hence have no effect upon the vertical reactions, which would be the same as for a simple truss or girder of the same span. To obtain the horizontal reactions it is necessary to make use of the equation of condition, viz., that the moment about the centre hinge of all the forces on *either* side of that hinge equals zero. The application of this equation is so simple as to need no explanation. The influence lines for the vertical and horizontal components of the reactions are given in Figs. 212 and 213 for the



Influence Line for V_L

FIG. 212.



Influence Line for H_R and H_L

FIG. 213.

arch shown in Fig. 211 in order to show clearly the variations in the reactions as the load crosses the structure. These lines are also correct for the spandrel-braced arch shown in Fig. 210, provided the arch rib has the same dimensions, since the construction above the arch rib has no influence upon the value of the reactions.

For a uniform load the maximum value of both the horizontal and vertical components, and hence of the actual reaction,

evidently occurs when the entire structure is loaded, while the maximum reaction for a concentrated load occurs when the load is placed at the centre of the span. In designing the piers it is as important to know the direction of the reaction as its magnitude. Both may be determined graphically for any position of the loads by the methods shown in Fig. 212, in which the sloping dotted lines showing the direction are determined by laying off at the foot of each vertical the corresponding horizontal ordinate as obtained by scale from Fig. 213. It will be observed that the direction of the left reaction is constant for loads on the right half of the structure. This is not accidental, but is due to the effect of the centre hinge. Since with a load on the right half of the structure the only force to the left of the centre hinge is the left reaction, and since the moment about the centre hinge equals zero, the left reaction must pass through it. This principle may be stated as follows: For a load to the *right* of the centre hinge the direction of the *left* reaction coincides with a line drawn through the left and centre hinges, and *vice versa*. It is evident that while the reaction at one end due to the live load on the other half of the arch may pass through the end and centre hinges, the actual reaction will not have this direction, since such a condition would involve the entire absence of dead load in the half of the structure adjoining the reaction in question.

With a concentrated-load system the maximum vertical reactions may evidently be determined as for any simple beam, while the shape of the influence line shows that the maximum horizontal component will occur for that position of the live loads which would give a maximum moment at the centre of a span of the length of the arch, and hence may be easily determined. The exact position for the maximum value of the reaction itself is less easily determined, but an equivalent uniform load may be used with safety to determine the actual maximum reaction.

131. Maximum Stresses in Elastic Arch Ribs. The maximum fibre stress at any section of the arch rib of a structure like that shown in Fig. 211 may be determined if the direction, point of application, and magnitude of the resultant force at the section are known. It is a well-known principle of mechanics that a force P applied at one of the principal axes OY of a cross-section of a straight elastic bar in the manner shown in Fig. 214 causes a

direct fibre stress at a distance c from the other principal axis OX, which may be expressed by the equation:

$$s=\frac{N}{A}\pm Nv\frac{c}{I},$$

in which N = normal component of the force P.
S = transverse component.
A = area of cross-section.
v = distance of point of application of force from the axis OX.
I = moment of inertia of cross-section about axis OX.

The shearing stress due to the transverse component, S, may for such a case be computed in the same manner as in an ordinary beam.

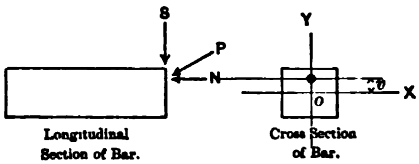


FIG. 214.

For a curved bar, such as an arch rib, the formula just given for the value of s is not strictly correct, but should be replaced by the following equation:

$$s=\frac{N}{A}\pm\frac{Nv}{AR}\pm Nv\frac{Rc}{R+c}\int\frac{R+y}{Ry^2dA},$$

in which R = radius of curvature of the axis of the arch.
y = distance of any fibre from axis OX.

For arches the radius of curvature, R, is always very large, compared with the dimensions of the cross-section, hence

$$\int\frac{R+y}{Ry^2dA} \text{ equals very nearly } \int\frac{1}{y^2dA}=\frac{1}{I}\cdot \quad\text{and}\quad \frac{Rc}{R+c}=c.$$

v is also small compared with R in any well-proportioned arch; hence the second term of above expression for s may be neglected with but little error, giving for a final value

$$s=\frac{N}{A}\pm Nv\frac{c}{I},$$

the same expression as for a straight bar.[1] In this formula Nv = external bending moment on the section, hence the formula may be written:

$$s=\frac{N}{A}\pm\frac{Mc}{I},$$

in which $M=Nv$.

In order to determine the maximum compression at the cross-section of any arch rib, it is necessary to determine the position of the loads which will produce the maximum value of the expression $\frac{N}{A}+\frac{Mc}{I}$, and to determine the maximum tension (or minimum compression), the position of loads giving the maximum negative value or minimum positive value of $\frac{N}{A}-\frac{Mc}{I}$ must be determined.

These equations are applicable for arches which can carry both tension and compression. If the arch can carry compression only, as in the case of the ordinary stone arch, they are correct only when the value of $\frac{N}{A}-\frac{Mc}{I}$ is positive. Masonry arches should, however, be so proportioned that this condition will always exist.

For uniform loads the simplest method of determining the position for maximum direct fibre stresses is by an influence table, in which the maximum values of the direct tension and compression at various sections for load unity at each panel point are given, a sufficient number of sections being chosen to ensure economy and safety in the design.

For arches carrying concentrated load systems, influence lines may be drawn for maximum stresses of both kinds at as many sections as may be desired, and the position of the loads determined in the manner previously used for trusses, or an influence table may be employed and the maximum stresses determined by trial, the value of the panel loads for probable positions being first tabulated. Examples of the computations for such an arch will not be given, as it involves nothing but the application of

[1] The error made by these approximations is extremely small, even for arches with as sharp a radius of curvature as an ordinary sewer arch. The general formula should, however, be employed in determining the stress in a curved bar such as a crane hook.

the principles already thoroughly illustrated for other structures, and the student who is familiar with these principles should have no difficulty in applying them to such a structure.

132. Parabolic Three-hinged Arches. In practice three-hinged arches are frequently constructed either with a parabolic axis or with panel points lying on a parabola. If the end pins of such an arch are at the same elevation, and if the load is vertical, uniformly distributed, and applied to the arch by vertical posts, the moment at any panel point equals zero.

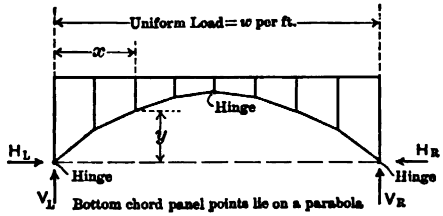


FIG. 215.

This proposition may be proven as follows:

Let H_L and V_L, Fig. 215, be the horizontal and vertical components respectively of the left reaction.

x and y equal the abscissa and ordinate respectively of any panel point on the arch axis referred to the left hinge.

M = moment at this point due to a uniform vertical load over the entire span.

M_v = moment at same point due to the vertical loads and vertical reactions only.

M_h = moment at the same point due to the horizontal reaction, H_L.

The vertical reactions in such a structure are the same as for an end-supported beam, therefore M_v equals the moment on such a beam, hence it varies as the ordinates to a parabola. (See Art. 43, Case 8.) Since H_L is constant for the loading under consideration, and y is the ordinate of a parabola, $M_h = (H_L)(y)$; therefore it also varies as the ordinates to a parabola. But $M = M_v - M_h$; therefore it also varies as the ordinates to a parabola, therefore $\frac{M}{y}$ is constant for every panel point. At the center

hinge $M=0$. $\therefore$ $\frac{M}{y}=0$ at this point and consequently at every panel point of the arch. For sections between panel points M_v varies as a straight line. (See Art. 36.) If the arch itself be straight between panel points, $(H_L)(y)$ also varies as a straight line, hence $\frac{M}{y}$ varies as a straight line between panel points and in consequence equals zero, hence the arch rib carries direct compression only. This is the ordinary condition for spandrel-braced arches, hence under a full uniform load the stresses equal zero in top chord and diagonals of such an arch, i.e., an arch conforming to the conditions stated at the beginning of this article; the stress in the verticals equals the panel load, and the stress in the bottom chord is direct compression throughout and has a horizontal component equal to the horizontal component of the reaction. If the arch rib be curved between panel points the bending moment in it will be zero at the panel points only.

For partial loads the moments at the panel points will not equal zero and the arch rib will be subjected to bending moments throughout its length. It should be observed, however, that the maximum positive moment at any panel point due to a uniform live load will equal the maximum negative moment at the same point due to the same load. This is due to the fact that the portion of the structure which should be loaded with a uniform load for maximum positive live moment at any section should be unloaded for a maximum negative live moment at the same section and *vice versa*, hence the combined loading for maximum positive and maximum negative moment is equivalent to a full uniform load, therefore the maximum positive live moment plus the maximum negative live moment equals zero.

For spandrel-braced arches a partial load causes stress in the diagonals and in all the top chord bars except the adjustable one, and the maximum tension in these members under uniform live load equals the maximum compression for the reasons already given. For this type of arch the bottom chord, or arch rib, carries only direct stress if straight between panel points, the structure acting like any other framed structure. With a concentrated load system the maximum positive bending moment will not equal the maximum negative moment, nor will they be equal for a uniform load with locomotive excess.

These conclusions for a spandrel-braced arch are confirmed by the problem which follows:

Problem. Compute the maximum stresses in all members of the spandrel-braced three-hinged parabolic arch shown in Fig. 216.

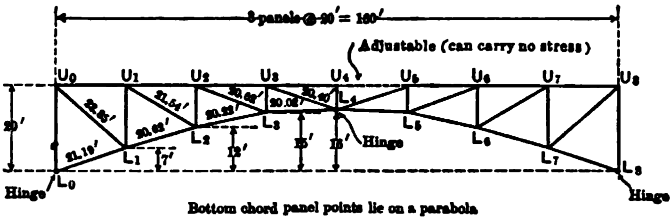


FIG. 216.

Dead weight of bridge,

1000 lbs. per foot per truss, top chord = 20,000 lbs. per panel.

400 lbs. per foot per truss, bottom chord = 8000 lbs. per panel.

Uniform live load:

2000 lbs. per foot per truss, top chord = 40,000 lbs. per panel.
Locomotive excess, 25,000 lbs.

This problem may be solved either by use of influence lines or an influence table. The latter will be employed here in order to illustrate its use.

The following laws concerning the magnitude and direction of the *left* reaction when the load is to the *right* of the centre hinge are of material assistance in preparing such a table.

a. The line of action of the left reaction passes through L_0 and L_4.

b. Its vertical and horizontal components and hence its magnitude varies directly with the distance of the load from U_8.

c. The moment about each of the bottom panel points to the left of the centre is counter-clockwise, hence the stress in the top chord is tension and that in the lower chord compression.

It follows from the above rules that the magnitude of the stress in all the lower chord bars in the left half of the arch varies uniformly as the load moves from U_8 to U_4, hence the magnitude of the stress in the web members of the left half of the arch also varies uniformly, since the stress in each of these members is a function of the combined vertical components of the left reaction and the stress in one of the bottom chord members. With the load on the *left* half of the arch the stresses in the bars on the left half of the structure will not vary uniformly, and may be either tension or compression, since the left reaction varies in magnitude and direction. The influence table will now be given, and a table giving maximum stresses in all bars follows.

INFLUENCE TABLE FOR REACTIONS AND HORIZONTAL COMPONENTS IN CHORDS

Load at	Reaction.		Horizontal Components of Stress in Chord Bars.					
	V_L	H_L	Bar.	H. C. Stress.	Bar.	H. C. Stress.	Bar.	H. C. Stress.
U_1	$\frac{7}{8}$	$\frac{1}{8}\times\frac{80}{16}=\frac{5}{8}$	U_0U_1	$-\frac{7}{8}\times\frac{20}{13}+\frac{5}{8}\times\frac{7}{13}$ $=-1.010$	U_1U_2	$-\frac{7}{8}\times\frac{40}{8}+\frac{5}{8}\times\frac{12}{8}+1\times\frac{20}{8}$ $=-0.938$	U_2U_3	$-\frac{7}{8}\times\frac{60}{5}+\frac{5}{8}\times\frac{15}{5}+1\times\frac{40}{5}$ $=-0.625$
			L_1L_2	$+\frac{7}{8}\times\frac{20}{13}-\frac{5}{8}\times\frac{20}{13}$ $=+0.385$	L_2L_3	$+\frac{7}{8}\times\frac{40}{8}-\frac{5}{8}\times\frac{20}{8}-1\times\frac{20}{8}$ $=+0.313$	L_3L_4	$+\frac{7}{8}\times\frac{60}{5}-\frac{5}{8}\times\frac{20}{5}-1\times\frac{40}{5}$ $=\ 0.000$
U_2	$\frac{3}{4}$	$\frac{1}{4}\times\frac{80}{16}=\frac{5}{4}$	U_0U_1	$-\frac{3}{4}\times\frac{20}{13}+\frac{5}{4}\times\frac{7}{13}$ $=-0.481$	U_1U_2	$-\frac{3}{4}\times\frac{40}{8}+\frac{5}{4}\times\frac{12}{8}$ $=-1.875$	U_2U_3	$-\frac{3}{4}\times\frac{60}{5}+\frac{5}{4}\times\frac{15}{5}+1\times\frac{20}{5}$ $=-1.250$
			L_1L_2	$+\frac{3}{4}\times\frac{20}{13}-\frac{5}{4}\times\frac{20}{13}$ $=-0.769$	L_2L_3	$+\frac{3}{4}\times\frac{40}{8}-\frac{5}{4}\times\frac{20}{8}$ $=+0.625$	L_3L_4	$+\frac{3}{4}\times\frac{60}{5}-\frac{5}{4}\times\frac{20}{5}-1\times\frac{20}{5}$ $=\ 0.000$
U_3	$\frac{5}{8}$	$\frac{3}{8}\times\frac{80}{16}=\frac{15}{8}$	U_0U_1	$-\frac{5}{8}\times\frac{20}{13}+\frac{15}{8}\times\frac{7}{13}$ $=+0.048$	U_1U_2	$-\frac{5}{8}\times\frac{40}{8}+\frac{15}{8}\times\frac{12}{8}$ $=-0.312$	U_2U_3	$-\frac{5}{8}\times\frac{60}{5}+\frac{15}{8}\times\frac{15}{5}$ $=-1.875$
			L_1L_2	$+\frac{5}{8}\times\frac{20}{13}-\frac{15}{8}\times\frac{20}{13}$ $=-1.923$	L_2L_3	$+\frac{5}{8}\times\frac{40}{8}-\frac{15}{8}\times\frac{20}{8}$ $=-1.562$	L_3L_4	$+\frac{5}{8}\times\frac{60}{5}-\frac{15}{8}\times\frac{20}{5}$ $=\ 0.000$
U_4	$\frac{1}{2}$	$\frac{1}{2}\times\frac{80}{16}=\frac{5}{2}$	U_0U_1	$-\frac{1}{2}\times\frac{20}{13}+\frac{5}{2}\times\frac{7}{13}$ $=+0.577$	U_1U_2	$-\frac{1}{2}\times\frac{40}{8}+\frac{5}{2}\times\frac{12}{8}$ $=+1.250$	U_2U_3	$-\frac{1}{2}\times\frac{60}{5}+\frac{5}{2}\times\frac{15}{5}$ $=+1.500$
			L_1L_2	$+\frac{1}{2}\times\frac{20}{13}-\frac{5}{2}\times\frac{20}{13}$ $=-3.077$	L_2L_3	$+\frac{1}{2}\times\frac{40}{8}-\frac{5}{2}\times\frac{20}{8}$ $=-3.750$	L_3L_4	$+\frac{1}{2}\times\frac{60}{5}-\frac{5}{2}\times\frac{20}{5}$ $=-4.000$

INFLUENCE TABLE FOR REACTIONS AND HORIZONTAL COMPONENTS IN CHORDS—*Continued*

Load at	Reactions.		Horizontal Components of Stress in Chord Bars.					
	V_L	H_L	Bar.	H. C. Stress.	Bar.	H. C. Stress.	Bar.	H. C. Stress.
U_5	$\frac{3}{8}$	$\frac{15}{8}$	U_0U_1	$+\frac{3}{4}\times 0.577 = +0.433$	U_1U_2	$+\frac{3}{4}\times 1.25 = +0.937$	U_2U_3	$+\frac{3}{4}\times 1.500 = +1.125$
			L_1L_2	$-\frac{3}{4}\times 3.077 = -2.308$	L_2L_3	$-\frac{3}{4}\times 3.750 = -2.812$	L_3L_4	$-\frac{3}{4}\times 4.000 = -3.000$
U_6	$\frac{1}{4}$	$\frac{5}{4}$	U_0U_1	$+\frac{1}{2}\times 0.577 = +0.289$	U_1U_2	$+\frac{1}{2}\times 1.25 = +0.625$	$U\ U_3$	$+\frac{1}{2}\times 1.500 = +0.750$
			L_1L_2	$-\frac{1}{2}\times 3.077 = -1.538$	L_2L_3	$-\frac{1}{2}\times 3.750 = -1.875$	L_3L_4	$-\frac{1}{2}\times 4.000 = -2.000$
U_7	$\frac{1}{8}$	$\frac{5}{8}$	U_0U_1	$+\frac{1}{4}\times 0.577 = +0.144$	U_1U_2	$+\frac{1}{4}\times 1.25 = +0.312$	U_2U_3	$+\frac{1}{4}\times 1.500 = +0.375$
			L_1L_2	$-\frac{1}{4}\times 3.077 = -0.769$	L_2L_3	$-\frac{1}{4}\times 3.750 = -0.938$	L_3L_4	$-\frac{1}{4}\times 4.000 = -1.000$

H. C. stress in $L_0L_1 = -H_L$ for each position of load.

Under full load the stress in any member equals the algebraic sum of the tabular stresses. For the top chord this should equal zero, and for the bottom chord it should equal the sum of the tabular values of H_L and should have a negative sign. The application of these tests shows the accuracy of the tabular values.

INFLUENCE TABLE FOR VERTICAL COMPONENTS IN DIAGONALS

V_1=shear in panel containing diagonal.
V_2=vertical component in bottom chord bar in panel as determined from previous table.
$V_3=V_1 \pm V_2$=vertical component in diagonal.

Load at	Bar U_0L_1.	Bar U_1L_2.	Bar U_2L_3.	Bar U_3L_4.
U_1	$V_1=+0.875$	-0.125	-0.125	-0.125
	$V_2=-0.219$	$+0.096$	$+0.047$	0.000
	$V_3=+0.656$	-0.029	-0.078	-0.125
U_2	$V_1=+0.750$	$+0.750$	-0.250	-0.250
	$V_2=-0.437$	-0.192	$+0.094$	0.000
	$V_3=+0.313$	$+0.558$	-0.156	-0.250
U_3	$V_1=+0.625$	$+0.625$	$+0.625$	-0.375
	$V_2=-0.657$	-0.481	-0.234	-0.000
	$V_3=-0.032$	$+0.144$	$+0.391$	-0.375
U_4	$V_1=+0.500$	$+0.500$	$+0.500$	$+0.500$
	$V_2=-0.875$	-0.769	-0.563	-0.200
	$V_3=-0.375$	-0.269	-0.063	$+0.300$
U_5	$V_3=-0.281$	-0.202	-0.047	$+0.225$
U_6	$V_3=-0.187$	-0.134	-0.031	$+0.150$
U_7	$V_3=-0.094$	-0.067	-0.016	$+0.075$

Under full load the vertical component in each diagonal equals the algebraic sum of the tabular values. This sum should and does equal 0, thus checking all the tabular values.

INFLUENCE TABLE FOR STRESSES IN VERTICALS

V_3=vertical component in diagonal running to joint at top of vertical. See previous table.

V_4=panel load at top of vertical.

V_5=stress in bar.

Load at	Bar U_0L_0.	Bar U_1L_1.	Bar U_2L_2.	Bar U_3L_3.	Bar U_4L_4.
U_0	V_3= .... V_4= −1.000 V_5= −1.000	0.000	0.000	0.000	0.000
U_1	V_3= +0.656 V_4= 0.000 V_5= −0.656*	−0.029 −1.000 −0.971	−0.078 0.000 +0.078	−0.125 0.000 +0.125	0.000
U_2	V_3= +0.313 V_4= 0.000 V_5= −0.313	+0.558 0.000 −0.558	−0.156 −1.000 −0.844	−0.250 0.000 +0.250	0.000
U_3	V_3= −0.032 V_4= 0.000 V_5= +0.032	+0.144 0.000 −0.144	+0.391 0.000 −0.391	−0.375 −1.000 −0.625	0.000
U_4	V_3= −0.375 V_4= 0.000 V_5= +0.375	−0.269 0.000 +0.269	−0.063 0.000 +0.063	+0.300 0.000 −0.300	1.000
U_5	V_5= +0.281	+0.202	+0.047	−0.225	0.000
U_6	V_5= +0.187	+0.134	+0.031	−0.150	0.000
U_7	V_5= +0.094	+0.067	+0.016	−0.075	0.000

* Note that (+) stress in diagonal gives (−) stress in vertical and that for full load the stress in each vertical equals unity.

INFLUENCE TABLE FOR STRESS IN EACH BAR FOR LOAD AT

Bar.	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7
U_0U_1	0.000	−1.010	−0.481	+0.048	+0.577	+0.433	+0.289	+0.144
U_1U_2	0.000	−0.938	−1.875	−0.312	+1.250	+0.937	+0.625	+0.312
U_2U_3	0.000	−0.625	−1.250	−1.875	+1.500	+1.125	+0.750	+0.375
U_3U_4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
L_0L_1	0.000	−0.662	−1.325	−1.990	−2.650	−1.990	−1.325	−0.662
L_1L_2	0.000	+0.397	−0.793	−1.980	−3.170	−2.380	−1.590	−0.793
L_2L_3	0.000	+0.317	+0.632	−1.580	−3.790	−2.840	−1.900	−0.950
L_3L_4	0.000	0.000	0.000	0.000	−4.000	−3.000	−2.000	−1.000
U_0L_1	0.000	+1.204	+0.575	−0.059	−0.688	−0.515	−0.343	−0.172
U_1L_2	0.000	−0.078	+1.500	+0.388	−0.725	−0.545	−0.362	−0.180
U_2L_3	0.000	−0.322	−0.644	+1.614	−0.260	−0.194	−0.128	−0.066
U_3L_4	0.000	−0.637	−1.275	−1.913	+1.530	+1.148	+0.765	+0.382
U_0L_0	−1.000	−0.656	−0.313	+0.032	+0.375	+0.281	+0.187	+0.094
U_1L_1	0.000	−0.971	−0.558	−0.144	+0.269	+0.202	+0.134	+0.067
U_2L_2	0.000	+0.078	−0.844	−0.391	+0.063	+0.047	+0.031	+0.016
U_3L_3	0.000	+0.125	+0.250	−0.625	−0.300	−0.225	−0.150	−0.075
U_4L_4	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000

The values in this table may be verified by the same methods used for preceding tables.

From the influence table for stress in each bar, the maximum live stresses, due to uniform load, may be easily obtained for any given bar by summing up the total positive and negative values for the bar and multiplying each sum by the live panel load. The stress due to the locomotive excess may be computed by multiplying the maximum value for each bar by the excess load.

The table which follows shows the stresses thus obtained:

TABLE FOR MAXIMUM LIVE STRESSES IN ALL BARS

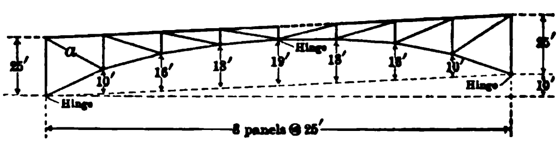
Bar.	Tension.			Compression.		
	Uniform Live at	E at	Stress, Lbs.	Uniform Live Load at	E at	Stress, Lbs.
U_0U_1	U_3 to U_7 inc.	U_4	74,100	U_1 and U_2	U_1	84,900
U_1U_2	U_4 to U_7 inc.	U_4	156,200	U_1 to U_3 inc.	U_2	171,900
U_2U_3	U_4 to U_7 inc.	U_4	187,500	U_1 to U_3 inc.	U_3	196,900
U_3U_4	....	.....	....	....	.....	....
L_0L_1	....	.....	....	U_1 to U_7 inc.	U_4	490,400
L_1L_2	U_1	U_1	25,800	U_2 to U_7 inc.	U_4	507,500
L_2L_3	U_1 and U_2	U_2	53,800	U_3 to U_7 inc.	U_4	537,200
L_3L_4	....	.....	....	U_4 to U_7 inc.	U_4	500,000
U_0L_1	U_1 and U_2	U_1	101,300	U_3 to U_7 inc.	U_4	88,300
U_1L_2	U_2 and U_3	U_2	113,100	U_1 and U_4 to U_7 inclusive	U_4	93,700
U_2L_3	U_3	U_3	104,900	U_1 and U_2 and U_4 to U_7 inclusive	U_2	80,700
U_3L_4	U_4 to U_7 inc.	U_4	191,300	U_1 to U_3 inc.	U_3	200,600
U_0L_0	U_3 to U_7 inc.	U_4	48,200	U_0 to U_2 inc.	U_0	83,800
U_1L_1	U_4 to U_7 inc.	U_4	33,600	U_1 to U_3 inc.	U_1	91,200
U_2L_2	U_1 and U_4 to U_7 inc.	U_1	11,400	U_2 and U_3	U_2	70,500
U_3L_3	U_1 and U_2	U_2	21,300	U_3 to U_7 inc.	U_3	70,700
U_4L_4	....	.....	....	U_4	U_4	65,000

The dead stresses are as follows:

Top chord bars and diagonals........................	0
End verticals..	10.000 lbs.
Intermediate verticals...............................	20,000 lbs.
Bottom chord bars (horizontal component).............	280,000 lbs.

PROBLEM

60. *a.* Draw influence line for horizontal components of reactions.
b. Draw influence line for vertical component of stress in bar *a*.



PROB. 60.

CHAPTER XI

DESIGN OF COLUMNS AND TENSION MEMBERS

133. Columns—General Considerations. A column is a member designed primarily to resist compression, although it may also be subjected to transverse loads causing flexure. For the present columns of the first type only will be considered. Compressive tests of blocks of plastic material of such proportions that the length does not greatly exceed the minimum lateral dimension show that failure occurs by lateral flowing of the material with no well-defined ultimate strength; a definite elastic limit, however, exists beyond which the material simply expands laterally and contracts longitudinally under increasing loads. On the other hand, compressive pieces in which the ratio of length to least lateral dimension is high, fail by lateral bending, when subjected to compression, even when the load is applied along the longitudinal axis passing through the centre of gravity of the bar, and the bar is originally straight and of homogeneous material without initial stress.

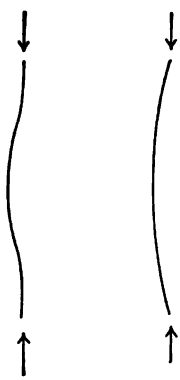


FIG. 217.

The ultimate load per square inch for the latter class of columns may be much less than the product of the elastic limit and the cross-section area. A good illustration of such a condition is presented by a straight bar of tempered steel of very small cross-section. A short piece of such a bar would sustain a high load per square inch without showing signs of failure, while a long piece would collapse by bending laterally under a comparatively light load, the column bending in one of the ways indicated by Fig. 217. The columns used in engineering structures generally have a slenderness ratio midway between these two extremes, hence failure may be expected either by crushing or bending, or by both together, even if the

column be originally in an ideal condition so far as material, shape, and loading are concerned.

The ideal column, however, does not exist in practice. The load is seldom if ever applied exactly at the centre of gravity or along the column axis; the process of fabrication in a metal column is sure to leave the column with some distortion, and with the material in a condition of initial stress, and columns of timber or concrete are equally sure to be imperfect. Moreover, the material is never homogeneous, and in a built-up steel column, such as is generally used in important structures, the behavior of the column as a whole is dependent upon the integrity of its cross-section, which may or may not be preserved by the rivets, tie-plates, lattice bars, and other devices required to hold together the main pieces.

In view of these many uncertainties, the economical and efficient design of columns is one of the most serious problems which the engineer has to confront, especially when dealing with unusual cases. The difficulties are heightened by the lack of sufficient experimental data. In recent years the study of the subject has, however, received a decided impetus, due in great part to the failure of the compression chords of the Quebec bridge, and many valuable data are being collected.

134. Condition of Ends. If the ends of a column are unrestrained against turning, it is said to have hinged ends; this condition, however, seldom exists. Columns in which the loads are applied by pins at the ends, as in many American bridges, are said to have pin ends. Columns in which the ends are subject to such restraint that the tangents to the elastic curve at the ends remain parallel to the column axis when the column deflects laterally are said to have fixed ends. If the ends of the column are square and bear upon flat surfaces, they are said to be square or flat-ended; this condition closely approximates the condition of fixed-ended columns when the columns are short, and pin-ended columns when long.

That the condition of the end may affect the strength of the column is apparent from a study of Fig. 218, which shows the curves which both round-ended and fixed-ended columns would take under vertical loads before failure by bending.

These two cases are somewhat analogous to free-ended and fixed-ended beams. A fixed-ended beam is materially stronger

than one with ends simply supported; and the same is true with columns. The portion *cd* of the fixed-ended column corresponds to the entire length of the round-ended column, the points *c* and *d* being points of contraflexure. The distance between *c* and *d* equals one-half of *ab*, since the portion *ce* of the column is in the same condition as *ac*; that is, the tangent to the elastic curve at *e* is parallel to the original axis of the column, and so also is the tangent at *a*. It follows that in comparing fixed-ended with round-ended columns it may be considered that the unsupported length in one case is half that of the other.

Columns with ends entirely free to turn do not exist in actual structures. The nearest approach to this condition probably occurs in the ordinary pin-ended column, but such pins are by no means frictionless; indeed, in some cases after exposure to weather, with the consequent rusting which takes place, the pins are so restrained that it is with great difficulty that the members can be turned about them. It is also seldom that structural columns are rigidly fixed at the ends, since the piece to which the column is riveted is seldom so rigid that it will not yield somewhat under the influence of the bending tendency. Owing to these facts the former practice of using column formulas based upon the end conditions has been abandoned by most American engineers.

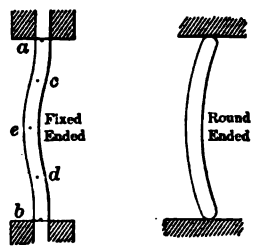


FIG. 218.

135. Formulas for Columns of Ordinary Lengths. No entirely satisfactory formula for proportioning columns of lengths such as are common in ordinary structures has yet been developed. Two types of formulas are, however, in common use, the Gordon (or Rankine) formula, and the straight-line formula. Of these the first rests upon an imperfect theoretical basis. The latter is purely empirical. These two types of formulas are as follows:

Let P = total allowable load on the column.

A = area of cross-section in square inches.

f_c = allowable compression per square inch in a short prism.

$\frac{L}{r}$ = maximum ratio of unsupported length to radius of gyration. (Note that L and r should both be expressed in the same units.)

c = an experimental constant.

Gordon formula: $$\frac{P}{A}=\frac{f_c}{1+\frac{1}{c}\left(\frac{L}{r}\right)^2}.$$

Straight-line formula: $$\frac{P}{A}=f_c-\frac{1}{c}\left(\frac{L}{r}\right).$$

The latter formula is of the first degree, and consequently gives a straight line for values of $\frac{P}{A}$. It is simpler to use than the Gordon formula, although tables giving the value of $\frac{P}{A}$ as determined by the Gordon formula for different ratios of $\frac{L}{r}$ are published in the various structural handbooks.

Of these two formulas the former is the older and has been more generally used by American engineers and is still favored by many. The growing recognition of the fact that the strength of a column is largely dependent upon accidental conditions, which cannot be expressed theoretically, such as initial stress, lack of straightness, and unhomogeneity of material, coupled with the fact that experiments fail to demonstrate that a formula of the Gordon type corresponds more closely to experimental results than the straight-line formula, has led to the adoption of the latter by many engineers, so that an examination of the bridge specifications of twenty-seven typical American railways shows that at the present time thirteen of these specify the straight-line formula.

136. Typical Formulas for Columns of Ordinary Lengths. The two formulas which follow are typical of the Gordon and the straight-line formulas and represent present day American practice for ordinary structures. In both the condition of the ends is ignored.

Formula from Massachusetts Railroad Commission's "Specifications for Bridges Carrying Electric Railways." For structural steel having a required ultimate strength of from 55,000 to 65,000 lbs. Live stress to be corrected for impact:

$$\frac{P}{A}=\frac{12,000}{1+\frac{1}{20,000}\left(\frac{L}{r}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Formula from American Railway Engineering and Maintenance of Way Association's "General Specifications for Steel Railway Bridges." For structural steel having an ultimate tensile strength of about 60,000 lbs. per square inch. Live stresses to be corrected for impact:

$$\frac{P}{A}=16,000-70\frac{L}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Both of these formulas are for main members[1] and should be used for columns for which the value of $\frac{L}{r}$ *does not exceed* 100.

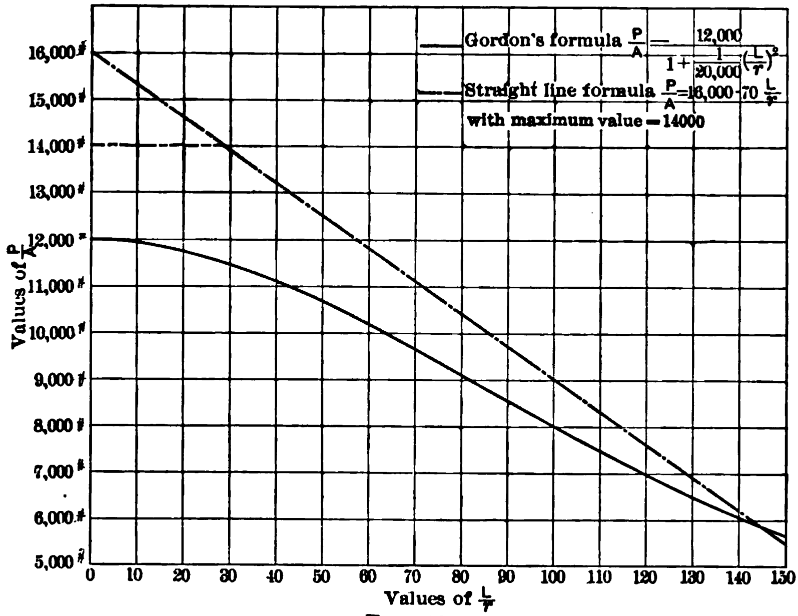


FIG. 219.

The term $\frac{L}{r}$ in these formulas should equal the maximum ratio of unrestrained length to radius of gyration. If the column is restrained against lateral deflection in all directions at the two ends and at no intermediate point, then L would be the

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[1] Secondary members, such as lateral struts, are commonly designed for a somewhat higher unit stress, and a larger value of $\frac{L}{r}$ is permissible. The following values may be used for such members:

$$\frac{L}{r} \leqslant 120. \quad \frac{P}{A}=1.20\left(16,000-70\frac{L}{r}\right).$$

total length of the column between lateral supports, and r the least radius of gyration of its transverse section, provided the column is of constant cross-section between supports, as is usually the case. If the column is held against lateral deflection in all directions at one or several intermediate points, the value of $\frac{L}{r}$ to be used should be the largest possible value for any portion of the column between any two points of support. If the column is held at an intermediate point in one direction only, then the value of $\frac{L}{r}$ to be used in the formula should be the maximum obtained by using for L either the length of any section or the total length of the column, and for r in each case the radius of gyration referred to the axis about which the column is free to bend. For example the maximum value of $\frac{L}{r}$ for the column shown in Fig. 220 may be either $\frac{L}{2r}$ or $\frac{L}{r_1}$.

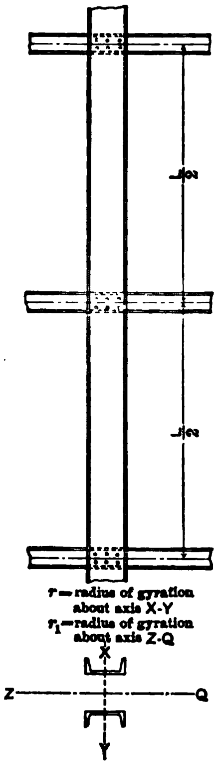


FIG. 220.

If the column is of variable section the designer must use his judgment in selecting the proper value of r to use.

137. Formulas for Long Columns. The term "long column," as used in this article, refers to columns of length such that failure tends to occur by lateral bending before the material has reached its elastic limit. The *collapsing load* which such columns will carry without yielding, when centrally loaded, can be closely determined by mathematical investigations,[1] provided the columns are initially straight and homogeneous, and the load axially applied, and is dependent upon the elasticity rather than the crushing strength of the material. The formula commonly used for such columns is known as the Euler formula, and is given in treatises on mechanics, as follows:

[1] See "Applied Mechanics," Lanza, Edition 9, pp. 330–333.

Columns with fixed ends: $\dfrac{P}{A}=4\pi^2E\left(\dfrac{r}{L}\right)^2$. . . (23)

Columns with round ends: $\dfrac{P}{A}=\pi^2E\left(\dfrac{r}{L}\right)^2$. . . (24)

Columns with one free end: $\dfrac{P}{A}=\dfrac{\pi^2E}{4}\left(\dfrac{r}{L}\right)^2$ . . (24a)

In these formulas E equals modulus of elasticity and $\frac{P}{A}$ is the axial stress required to hold the column in equilibrium if slightly deflected laterally. If a load greater than P be applied to such a column, the column will collapse; if a smaller load, the column will spring back to its original condition when the lateral forces are removed. If Euler's formula be applied to columns composed of material with an elastic limit of 33,000 lbs. and a modulus of elasticity of 29,000,000, both of which are reasonable values for structural steel, the value of $\frac{P}{A}$ for round-ended columns would exceed the elastic limit whenever $\frac{L}{r}<94$, hence Euler's formula should not be used for such columns when the ratio of length to radius of gyration is less than this limit. For fixed-ended columns, on the other hand, the use of the same constants would give for $\frac{L}{r}$ the value of 186.

In using these formulas a suitable factor of safety should be employed.

Since columns as used in structures are in an intermediate condition between round-ended and fixed-ended, it would seem that for columns for which this ratio exceeds about 150, Euler's formula should be used. The fact that the value of $\frac{L}{r}$ is usually restricted in bridges to a ratio of 100 is apparently a conservative custom and agrees with the column formula commonly employed.

138. Tests of Steel Columns. There are comparatively few carefully conducted tests of well-proportioned full-sized steel columns available for study. Among the most complete and reliable of these tests may be cited those made upon the 2,000,000 lb. testing machine of the Phœnix Iron Company, at Phœnixville, Pa., and conducted by James E. Howard. The results of these

tests were published in the "Proceedings of the American Society of Civil Engineers" for February, 1911, and show an ultimate strength of 30,000 lbs. per square inch for columns having a ratio of $\frac{L}{r}$ of 47.1 and composed of plates and angles the elastic limit of which varied in the different pieces from 29,400 to 37,300 lbs. per square inch, the former value occurring in one of the plates and being the minimum value found. The conclusion reached by Mr. Howard from these tests is that "the minimum value of the elastic limit, as found in the component parts, chiefly modifies the ultimate resistance of the columns, and that variations of its value would overshadow the considerations which find expression in empirical formulas for strength and take no account of such features."

The entire subject of column strength is now being investigated by a committee of the American Society of Civil Engineers, and it may be hoped that before long sufficient reliable data will be available to permit the establishment of better formulas with more reliable constants.

139. Cast-iron Columns. Cast iron is unsuitable for structural members exposed to tension or bending because of its low tensile strength and brittleness. It may, however, be used for compression pieces if these are properly designed, cast-iron columns being frequently used for interior columns in buildings. Such columns cannot be made in long lengths and the different sections cannot be fastened as rigidly together as steel columns, hence they are decidedly inferior in rigidity to the latter; moreover, the fact that it is difficult to secure uniform thickness of shell and that the material is often very variable in composition, may contain flaws, and is frequently in a state of initial stress, is against their use. It is also difficult to obtain good connections of transverse beams and girders.

A set of very valuable tests was made upon cast-iron columns at the works of the Phœnix Iron Company in 1896–97, and a formula based upon these tests is probably as reliable as anything that can be obtained, although the results of the tests were so variable that a large factor of safety should be used in applying the formula. A study of the tests shows that a straight-line formula conforms to the results, as well as any other type of formula. Such a formula is derived by Professor Wm. H. Burr

in "The Elasticity and Resistance of the Materials of Engineering," and is as follows:

Ultimate strength per square inch for circular flat-ended columns:

$$\frac{P}{A}=30{,}500-160\frac{L}{d}. \qquad (25)$$

In this formula L = unsupported length and d = diameter.

A formula based upon this same set of tests, as given by Johnson in "The Materials of Construction," differs somewhat from the above and is as follows:

$$\frac{P}{A}=34{,}000-88\frac{L}{r}. \qquad (26)$$

In this formula r = least radius of gyration.

A factor of safety of five, which is none too much for a material as uncertain as cast iron, reduces Burr's formula to the following form:

$$\frac{P}{A}=6100-32\frac{L}{d}. \qquad (27)$$

This value is very much less than the corresponding value for steel columns in spite of the high compression strength of cast iron, and in consequence, while cast iron is cheaper per pound than steel, there is little if any economy in the use of properly designed cast-iron columns except for light loads for which it may be difficult to obtain steel columns of sufficiently small cross-section. It should be noted, however, that the building laws of the large cities permit, in general, the use of much higher unit stresses than those given by either of the above formulas when properly reduced by a factor of safety, and that the use of cast-iron columns in buildings will probably continue until the legal unit stresses are reduced. The employment of cast iron in bridges was abandoned many years ago both because of its treacherous character and the difficulty of making satisfactory connections between members.

The limiting lengths to which these formulas are applicable are stated to be as follows:

Johnson: $\frac{L}{r} \leq 120$ r = radius of gyration.

Burr: $\frac{L}{d} \leq 40$ d = diameter of circular column or shorter side of rectangular column.

140. Timber and Concrete Columns. Timber columns resemble cast-iron columns in being very variable in strength. This is largely due to the presence of knots and other defects. Wide variations in the results of tests are noticeable, hence a straight-line formula is probably as well adapted to such columns as any other, and that given in Art. 18 may be used.

It is important to note that timber columns made by bolting a number of sticks together are no stronger than if each stick were to be separate and loaded by its share of the total load. This has been shown by tests and may be readily understood, since the bolts cannot be counted upon as holding the individual sticks in place, owing to the small bearing value of wood across the grain and the difficulty of keeping nuts tight.

Concrete columns will not be considered here. The student is referred to books upon concrete structures such as "Concrete Plain and Reinforced," by Taylor and Thompson, and "Principles of Reinforced Concrete," by Turneaure and Maurer, for full treatment of such columns.

141. Typical Column Sections. Fig. 221 represents the cross-sections of a number of types of columns. *A* and *B* are columns frequently used in bridge construction, the latter set representing the common type for upper chords of pin bridges, the horizontal plate on the top flange being used to give lateral rigidity. *C* shows some very heavy column sections, used in the Queensboro Bridge, the Metropolitan Tower, and the Bankers' Trust Building, all in New York city. *D* shows columns sometimes used in elevated railroad construction in which the central diaphragm is useful both in adding to the cross-section and in preserving the integrity of the column. *E* is a type of column frequently used for verticals of riveted trusses. *F* is a Z-bar column much used in building construction. *G* is the well-known Phœnix column, made by the Phœnix Iron Company and once widely used for bridges and elevated railroads. *L* is the Larimer column made by Jones & Laughlins, Limited.

H shows H-section columns made by the Bethlehem Steel Company. *I* is an ordinary I-beam column; and *J* is an angle column.

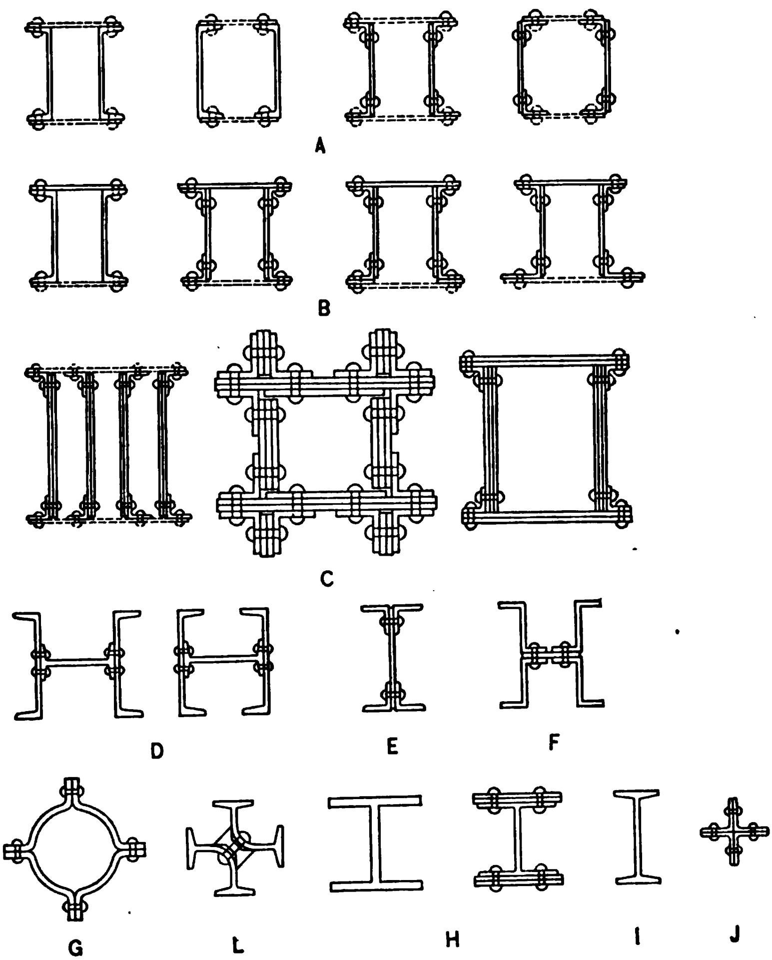


FIG. 221.—Column Types. Dotted Lines Represent Latticing.

142. General Dimensions and Limiting Conditions. In designing a column the first thing to be determined is the type and general dimensions of the member; that is, the width and depth, provided these are limited by other considerations than those of strength, as is often the case. For example, the com-

pression chords in bridges must be of sufficient width and depth to permit of proper connections to the web members, and the verticals must be of such a size as to give suitable floor-beam connections; in buildings the columns are frequently limited in size because of the space available or the character of the necessary connections. Ease of construction must also be considered, and this is frequently a ruling factor, as in the case of channel columns with flanges turned toward each other, where the space between the flanges should not be less than 4 ins. and lattice bars[1] must be far enough apart to permit insertion of hand. This is illustrated in Fig. 222.

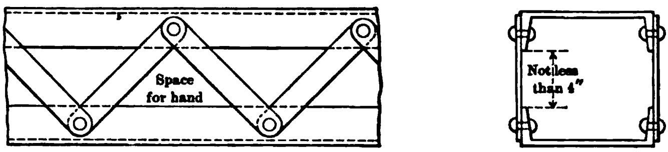


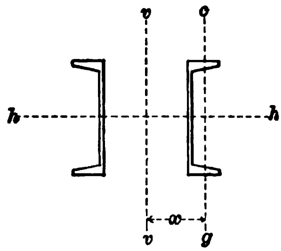
Fig. 222.

In determining the dimensions of the individual pieces there are also restrictions due to practical considerations. If the web or cover plates are too thin they may wrinkle under compression, hence it is common practice to limit the thickness of webs and cover plates to not less than from $\frac{1}{30}$ to $\frac{1}{40}$ of the distance between connecting rivets. It is also desirable to so proportion the column that the centre of gravity will be near the centre of section. If a cover plate is used on the top flange as in the chords of Fig. 221 *B*, unequal-legged angles with wide leg horizontal, or narrow flange plates, vertical or horizontal, are often used on the bottom flange to lower the centre of gravity.

143. Method of Design. With these restrictions in mind, an approximate design of the column may be made, either by assuming the value of the minimum radius of gyration or of the allowable unit stress. The actual allowable unit stress for the section thus obtained may then be computed, and the column redesigned if this stress varies too widely from the stress which the column actually carries. It is usually economical to place the

[1] Lattice bars are diagonal members such as shown in Fig. 222 intended to hold the two column halves in line and make the column act as a solid piece. These bars are of great importance and will be fully treated later.

webs such a distance apart that the radius of gyration about the principal axis parallel to the webs will equal that about the principal axis perpendicular to the webs, that is, if the unsupported length of the column is the same with respect to both axes. This involves the condition that the moments of inertia about each of these axes should be equal, since the radii of gyration will then be equal, and the value of $\frac{L}{r}$ and consequently of the allowable unit stress will be the same in both directions. The following method of accomplishing this for a column composed of two channels as shown in Fig. 223 is simple, and illustrates the problem sufficiently.



Axis $c.g.$ passes through centre of gravity of one channel and axes hh and vv through centre of gravity of column.

FIG. 223.

Let r_h = radius of gyration about axis hh. (This is unalterable for any given channel.)

r_v = radius of gyration about axis vv.

I_v = moment of inertia of cross-section about axis vv.

I_h = moment of inertia of cross-section about axis hh.

I_{cg} = moment of inertia of each channel about its axis cg.

A = area of one channel.

To determine the moment of inertia about axis vv the following principle of mechanics may be used:

The moment of inertia of a section about any axis equals the moment of inertia of that section about a parallel axis passing through its centre of gravity, plus its area multiplied by the square of the distance between the two axes.

From the application of this principle the following equation results:

$$I_v = 2(I_{cg} + Ax^2).$$

Hence I_h should equal $2(I_{cg} + Ax^2)$.

But $\quad I_h = 2(Ar_h^2)$ hence $2Ar_h^2 = 2I_{cg} + 2Ax^2$.

Hence $$x^2 = \frac{2Ar_h^2 - 2I_{cg}}{2A} = r_h^2 - \frac{I_{cg}}{A}.$$

In the case of channel columns the value of I_{cg} is usually small compared with A, hence the error involved in omitting the last term of the equation is small and is on the safe side, therefore it may be neglected and the value of x made equal to r_h.

The proper distance between channels to secure equal rigidity about either axis is given in some of the steel manufacturers' handbooks and need not be computed; but for more complicated sections, such as plate and angle columns, it must usually be determined in the manner indicated, although the approximation mentioned is not always allowable, and, in the case of top chords with cover plates would be considerably in error and should not be made.

144. Determination of Cross-section of Typical Steel Columns.

Problem. Design a channel column for the following assumed conditions:

Total applied load (live, dead, and impact) = 250,000 lbs.
Unsupported length = 25 ft.
Allowable stress
$$\frac{P}{A}=\frac{16,000}{1+\frac{1}{20,000}\left(\frac{L}{r}\right)^2}.$$

Solution. Determine trial section by assuming $\frac{P}{A}=14,000$ lbs. This gives a trial area of $\frac{250,000}{14,000}=17.9$ sq.ins., which could be obtained by the use of two 15-inch channels at 33 lbs., having a total area of 19.8 sq.ins.

The radius of gyration for this section about an axis perpendicular to the web = 5.62 ins., hence the allowable value of

$$\frac{P}{A}=\frac{16,000}{1+\frac{25^2\times12^2}{20,000\times5.62^2}}=14,000 \text{ lbs.}$$

The actual unit stress if the proposed section should be used equals $\frac{250,000}{19.8}=12,600$ lbs. It follows that the column can stand considerably more than the applied load, hence it is safe and may possibly be decreased in size.

The next smaller channel is 12 in., 30 lb., hence try this. The allowable value of $\frac{P}{A}$ for a column composed of these channels equals

$$\frac{16,000}{1+\frac{25^2\times12^2}{20,000\times4.28^2}}=12,800 \text{ lbs.}$$

The actual unit stress, if these channels be used equals $\frac{250,000}{17.64}=$ 14,200 lbs.; hence these channels are too small, and the 15-inch channels should be chosen. These should be placed so that the distance x in Fig. 223 = 5.62 ins. "Cambria Steel" gives the proper distance between the webs as 9.51 ins., which agrees closely with the corresponding value when $x = 5.62$ ins.

Problem. Design a top-chord member of a bridge using a top cover plate:

Minimum clear distance between web plates	= 10 ins.
Minimum clear depth	= 17 ins.
Total applied load (live, dead, and impact)	= 430,000 lbs.
Unsupported length	= 25 ft.

Column formula, $\frac{P}{A}=16,000-70\frac{L}{r}$.

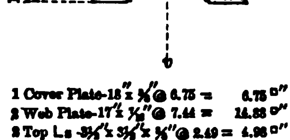
Thickness of web to be not less than $\frac{1}{30}$ distance between horizontal flange rivets.

Thickness of cover plates to be not less than $\frac{1}{40}$ distance between vertical flange rivets.

Solution. In a compression piece of this type, r, with respect to the horizontal axis, is usually about $\frac{4}{10}$ the depth of the member, therefore this value for the radius of gyration will be tried instead of assuming the allowable unit stress, as was done in the previous example. Making this assumption, and assuming also the minimum depth of 17 ins., and a distance apart of webs of 10 ins., gives 6.8 ins. as the trial value of r. Substituting this value in the column formula gives

$$\frac{P}{A}=16,000-70\frac{25\times 12}{6.8}=13,000 \text{ lbs.}$$

Using this value gives $\frac{430,000}{13,000}=33$ sq.ins. as the necessary area for a preliminary trial.



1 Cover Plate-18" x 3/8" @ 6.75 = 6.75 □"
2 Web Plate-17" x 7/16" @ 7.44 = 14.88 □"
2 Top Ls -3½" x 3½" x 3/8" @ 2.49 = 4.98 □"
2 Bottom Ls-6" x 3½" x 3/8" @ 3.43 = 6.86 □"
33.47 □"

FIG. 224.

The section shown in Fig. 224 complies with all the restrictions stated, and has an area somewhat greater than that just determined.

Bottom angles with a wider horizontal leg than the top angles are chosen in order to partially offset the effect of the top cover plate and thus lower the centre of gravity of the cross-section.

The exact value of r must now be computed, both about the axis vv and the axis hh, and the smaller value used to determine the allowable unit stress. The computations may conveniently be arranged in the tabular

form which follows, and which requires no explanation. It should be observed that the position of the centre of gravity is determined as a step in the process of finding the moment of inertia. It is necessary to locate its position in order to detail the structure properly, since the centre of gravity lines of the various members meeting at any joint should intersect at a point.

COMPUTATION OF I_{hh}

Piece.	Area.	Lever Arm.	Moment about *cl.*	Moment about *cl.*	*I* about *cl.*
Cover plate.....	6.75	+8.80″	+59.4	....	$59.4 \times 8.80 = 523$
Webs..........	14.88	0	0.0	....	$\frac{1}{12} \times \frac{7}{8} \times 17^3 = 358$
Top angles......	4.98	+7.61″	+37.8	....	$37.8 \times 7.61 = 287$
Bottom angles ..	6.86	−7.83	....	−53.7	$53.7 \times 7.83 = 420$
Total........	33.47	....	+97.2	−53.7	1588

$$z = \frac{97.2 - 53.7}{33.47} = 1.30 \text{ ins.} \qquad I_{hh} = 1588 - 33.47 \times 1.30^2 = 1531,$$

$$r_{hh} = \sqrt{\frac{1531}{33.47}} = 6.77 \text{ ins.}$$

COMPUTATION OF I_{vv}

Piece.	Area.	Lever Arm.	*I* about *vv.*
Cover......................	6.75	....	$\frac{1}{12} \times \frac{3}{8} \times 18^3 = 182$
Webs......................	14.88	5.22	$14.88 \times 5.22^2 = 405$
Top angles..................	4.98	6.45	$4.98 \times 6.45^2 = 207$
Bottom angles...............	6.86	7.47	$6.86 \times 7.47^2 = 383$
Total......................	33.47	....	$I_{vv} = 1177$

$$r_{vv} = \sqrt{\frac{1177}{33.47}} = 5.93 \text{ ins.}$$

The minimum value of r for the assumed section is evidently that about axis vv. The allowable value of $\frac{P}{A}$ for this case equals 12,460 lbs. The actual stress on the section would equal $\frac{430,000}{33.47} = 12,800$ lbs. per square inch, hence the area is slightly too small and should be increased, or else the webs should be placed sufficiently far apart to increase the value of r enough to give a proper allowable unit stress. The value of r about axis hh need not be increased, since it is almost

equal to the assumed value, and the area is larger than needed for that value.

These examples serve to illustrate the computations necessary for any form of "built-up" steel column. In many cases, however, the value of r can be taken directly from a handbook, e.g., in "Cambria Steel" are tables giving this value for Z-bar columns, for channel columns with cover plates, and for other sections.

145. Lattice Bars and Batten Plates. If the two ribs of a column such as that shown in cross-section by Fig. 223 be not connected, each rib would have to be proportioned as a separate column subjected to one-half the total load. The least radius of gyration for such a case would be that for one rib about the axis *cg*, which would ordinarily be much less than the value about axis *hh*, and consequently much smaller than the maximum value attainable for the sections used. Such a design would require a much larger amount of material for the main section than would be necessary if the two ribs should be rigidly connected so that they would act together, and the extra amount of material required would be much in excess of that needed for the details necessary to so connect the two ribs. Several conventional methods exist of connecting the ribs, the use of side plates or diaphragms, as illustrated by several of the cases of Fig. 221, being the most obvious. Either of these methods has the advantage of using for this purpose material which can also carry a portion of the stress. For bridge trusses in which connection by pins or field rivets must ordinarily be made to the side ribs, the use of plates on all four sides throughout the length of the column is, however, impracticable, and is also subject to the further disadvantage of giving a closed section which cannot be inspected for corrosion after erection, and the interior of which cannot be painted. The use of a diaphragm is also frequently impracticable for bridge members, owing to difficulties of designing proper details; and the same difficulty applies to the one-web columns, shown in Fig. 221. Moreover, it is desirable to have as much of the material as possible concentrated in the ribs, since the distribution of the stress over the cross-section is thereby simplified. For such columns, it is therefore common to connect the two ribs by short plates, usually called "batten plates" or "tie plates," at each end and at points where the continuity of the

latticing is interrupted, and to use diagonal bars throughout the remainder of the column, thus connecting the two ribs by a form of trussing. Such a column is shown in Fig. 225, in which latticing is used on both sides. It is frequently possible to use plates on one side of the column, as in the top chords shown in Fig. 221, *B*, in which case latticing should be employed on the other side. The latticing may be composed of flat bars, angles, or even small channels for unusually heavy columns, and may be single on each side, as shown in Fig. 225, or double, with rivets at the points of intersection, as shown by Fig. 226. The fact that the strength of latticed columns is largely dependent upon the proper design of the latticing requires that the proportioning of the latticing should be as carefully studied as the design of the main cross-section. Unfortunately the theoretical treatment of such details is more obscure than that of the columns themselves. It is evident, however, that if the column were to remain absolutely

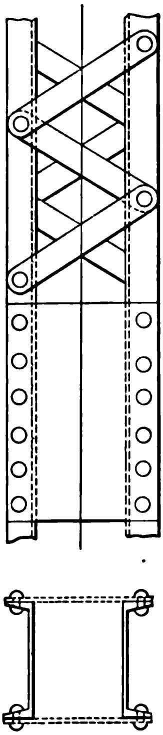


FIG. 225.

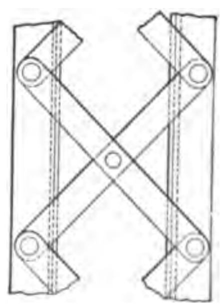


FIG. 226.

straight under loading, no latticing would be needed, and the stress in such lattice bars as might be used would be merely the secondary stress due to the shortening of the column as a whole and the consequent distortion of the lattice bars. On the other hand, if the column bends somewhat under loading, as would probably be the case unless the column were of very short length, bending moment, and consequently transverse shear would

occur, which would cause stresses in the lattice bars, and if the value of this shear can be determined the lattice bars may be easily proportioned. The magnitude of the bending moment in ordinary columns of the limited lengths consistent with good practice is largely dependent upon the unintentional eccentricity of the load due to the initial condition of the column with respect to its straightness, initial stress, or homogeneity of material, and cannot be determined. It may be estimated, however, to conform to the conditions assumed in proportioning the column, thereby securing a consistent design, by determining the external shear equivalent to the bending moment which causes the fibre stress in such a column to exceed the fibre stress in a short prism, and assuming the lattice bars to act as web members of a truss subjected to this bending moment. Such a method, while approximate, is perhaps as accurate as any yet derived, and will be developed to correspond to the bending moment obtained by the straight-line formula. For other formulas a similar method may be adopted.

Let f= fibre stress due to bending;
w= an assumed load uniformly distributed and applied at right angles to the column axis;
c= distance from neutral axis to extreme fibre of column;
M= external bending moment due to load w;
S= maximum external shear;
A= area of cross section;
I= moment of inertia of cross section about proper axis;
r= radius of gyration corresponding to I;
L= unsupported length of column.

Assume that the bending moment in the column equals that which would occur if the column were loaded uniformly at right angles to its axis throughout its length by the load w per foot, this giving a larger shear than would occur with any other reasonable assumption, such, for example, as a concentrated load applied at the centre.

The assumed distribution of stress over the cross-section of the column corresponding to the value given by the column formula is shown by Fig. 227, from which it is evident that

$$f = 16{,}000 - \frac{P}{A}.$$

But $$\frac{P}{A}=16{,}000-\frac{70L}{r} \qquad \therefore\ f=\frac{70L}{r}.$$

Also $$f=\frac{Mc}{I}=\left(\frac{1}{8}wL^2\right)\frac{c}{I}=\frac{1}{8}\frac{wL^2c}{Ar^2}.$$

$$\therefore\ \frac{1}{8}\frac{wL^2c}{Ar^2}=\frac{70L}{r}.$$

Hence $$w=\left(\frac{560}{L}\right)\left(\frac{Ar}{c}\right) \quad \text{and} \quad S=\frac{wL}{2}=280\left(\frac{Ar}{c}\right).$$

The total stress in a lattice bar, if single latticing is used, may now be taken as equal to one-half the product of S and the cosecant of the angle which it makes with the longitudinal axis of the column. This method gives the stress in the end lattice bars, but it is common to use the same size bars throughout the column. The following problem illustrates this method:

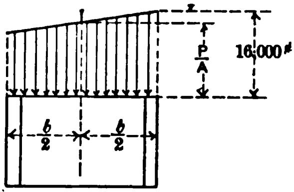


FIG. 227.

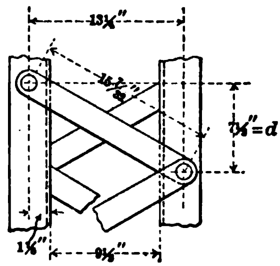


FIG. 228.

Problem. Determine stress in lattice bars for the 15-in. 33 lb. channel column designed in Art. 144.

Solution. For this column $A=19.8$ sq.ins.

$$r=5.62 \text{ ins.}$$

$$c=\frac{9.5}{2}+3.4=8.15 \text{ ins.}$$

hence $$S=\frac{280\times19.8\times5.62}{8.15}=3830 \text{ lbs.}$$

If the column is single latticed, as shown in Fig. 228, this shear will be equally divided between two bars, and the actual stress in each bar will be $\frac{3830}{2}\times\frac{15.22}{13.25}=2180$ lbs. A thickness of $\frac{1}{40}$ the distance between rivets, would require these bars to be 0.38 in. thick, hence $\frac{3}{8}$-in. bars would be sufficient to comply with this condition. A width of $2\frac{1}{2}$ ins. would commonly be adopted for such a column, hence the stress in the bar would equal $\frac{2180}{2\frac{1}{2}\times\frac{3}{8}}=2330$ lbs. per square inch.

The value of the radius of gyration for a rectangular bar of width b and thickness $t = \sqrt{\frac{bt^3}{12bt}} = t\sqrt{\frac{1}{12}} = 0.288t$, hence $\frac{L}{r}$ for such a bar $= \frac{15.22 \times 8}{0.288 \times 3} = 140$ approximately. If the straight-line formula be applied the allowable unit stress would be $\frac{P}{A} = 16{,}000 - 9800 = 6200$ lbs. The secondary stress in bars of such flat slope would not be large, but will be computed in order that its effect may be seen.

The direct stress in the channels $= 12{,}600$ lbs. per square inch, hence the reduction in the distance d under load would be $\frac{7.5 \times 12{,}600}{30{,}000{,}000} = 0.00315$ in., therefore the length of the lattice bar would be decreased by the following amount:

$$\sqrt{(13\tfrac{1}{4})^2 + (7\tfrac{1}{2})^2} - \sqrt{(13\tfrac{1}{4})^2 + 7.49685^2} = 0.001550 \text{ in.},$$

which corresponds to a stress of 3100 lbs. per square inch. The maximum stress, including secondary stress, therefore equals 5430 lbs. per square inch, which is well within the allowable limit. If an allowable unit stress in the lattice bars somewhat higher than that for the main section be considered permissible, the stress in the bars will be still more on the safe side.*

If the load be intentionally eccentric, as in the columns treated later, the same general method may be adopted, the excess fibre stress and shear corresponding to it being obtained from the formula given later for such columns. In addition to designing the lattice bars to carry the stress determined by this method it is common to impose certain arbitrary conditions as to size of bars and rivets. The following clauses from the "General Specifications for Steel Railway Bridges," published by the American Railway Engineering and Maintenance of Way Association, are typical of such restrictions.

"The minimum width of lattice bars shall be $2\frac{1}{2}$ ins. for $\frac{7}{8}$-in. rivets, $2\frac{1}{4}$ ins. for $\frac{3}{4}$-in. rivets, and 2 ins. if $\frac{5}{8}$-in. rivets are used. The thickness shall not be less than one-fortieth of the distance between end rivets for single lattice, and one-sixtieth for double lattice. Shapes of equivalent strength may be used.

"Five-eighths inch rivets shall be used for latticing flanges less than $2\frac{1}{2}$ ins. wide, and $\frac{3}{4}$-in. for flanges from $2\frac{1}{2}$ to $3\frac{1}{2}$ ins. wide; $\frac{7}{8}$-in. rivets shall be used in flanges $3\frac{1}{2}$ ins. and over, and lattice bars with at least two rivets shall be used for flanges over 5 ins. wide.

* N. Y. Central R. R. specifications require that columns must carry a transverse shear equal to about 2% of axial load.

"The inclination of lattice bars with the axis of the member shall not be less than 45 degrees, and when the distance between rivet lines in the flanges is more than 15 ins., if single riveted bars are used, the lattice shall be double and riveted at the intersection."

The tie plates at the ends or other points are usually proportioned by empirical methods. The common rule for tie plates on main members is to make the length of end plates not less than the distance between the lines of rivets connecting them to the flanges, and intermediate plates not less than one-half this length. Their thickness should not be less than one-fiftieth this distance.

It is to be hoped that the thorough investigation of steel columns now being conducted may throw further light on the subject of proportioning lattice bars and other column details, the importance of which in developing the full strength of the columns cannot be overestimated.

In a latticed column it is evidently essential that each rib between points of connection of the lattice bars shall be strong enough as a column to carry its share of the total load, hence the distance apart of the lattice bars when measured along the rib should be such that $\frac{L}{r}$ for the rib, L being taken as the distance between latticing rivets, should be no larger than the corresponding term for the whole column; this, however, is seldom a limiting factor in the design of the latticing, the empirical rule as to maximum slope of the lattice bars being usually sufficient to cover this point.

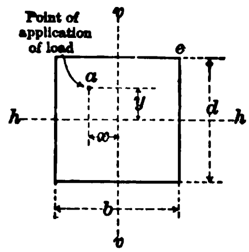
In connection with this subject it should be said that the columns of the famous Forth Bridge, the longest span bridge in the world, are of circular section, thus requiring no lattice bars or diaphragms and forming an ideal section so far as strength is concerned. These columns, however, were built in position, a method entirely opposed to American practice, in which the columns are built in the shops of the fabricating company and shipped intact to the bridge site, a method which limits the size of the column.

146. Rivet Pitch. The rivet pitch in "built-up" columns should be small enough to insure that wrinkling of the different parts between the rivets should not occur, and to properly

distribute the stress throughout the cross-section at the ends and at intermediate points where concentrated loads may be applied. The common rule is to use no spacing along the column axis greater than 6 ins. or 16 times the thickness of the thinnest connected piece, and to use at the ends and other points of application of the load a maximum pitch of four times the diameter of the rivet for a length equal to one and one-half times the maximum width of the member. If the bending moment carried by the column is large, as may be the case if loads of considerable eccentricity are applied, the rivet pitch should be investigated by the methods used for plate girders.

147. Eccentric Forces. If the resultant stress on any cross-section of a bar does not pass through its centre of gravity, the force is said to be eccentric. The effect of eccentric application of the load is to subject the section to a combination of direct stress and bending moment and to cause a maximum fibre stress considerably greater than would otherwise be the case. Such a loading should be avoided if possible. A similar condition arises if the resultant force on the cross-section is due to a direct force acting at the centre of gravity and a bending moment due to transverse flexure instead of eccentricity; and the two cases may be treated in the same manner.

General equations for the fibre stress at any point of a cross-section of any shape due to a combination of direct stress and bending moment are complicated and will not be given here, the reader being referred for a complete treatment of the subject to a paper by Professor Lewis J. Johnson in the Transactions of the Am. Soc. C. E., Vol. LVI, June, 1906.



Axes vv and hh are principal axes

FIG. 229.

The usual problem, that of determining the extreme fibre stress on a symmetrical cross-section of a straight bar, may be accomplished as follows: Consider first a straight bar subjected to a resultant thrust, acting parallel to its axis but not applied at the centre of gravity of the cross-section; and consider the bar to be so short that column action may be disregarded. Let the cross-section and point of application of the load be as shown in Fig. 229.

Let V = the vertical component in lbs. of a resultant thrust acting at point a.

A = area of cross-section in sq.ins.

I_h = moment of inertia of cross-section about axis hh.

I_v = moment of inertia of cross-section about axis vv.

f = compressive fibre stress at any corner (extreme fibre with respect to both axes).

Then $$f=\frac{V}{A}\pm\frac{Vxb}{2I_v}\pm\frac{Vyd}{2I_h}.$$

The last two terms of this equation give the fibre stress due to the bending moment resulting from the eccentric application of the load. If the piece be subjected to a direct axial thrust and transverse loads, the same equation would apply, but Vx and Vy would have to be replaced by M_h and M_v respectively, the bending moments due to transverse loads acting in planes hh and vv, respectively.

The proper sign to use for the last two terms may be determined from the character of the bending moment for the corner under consideration, with respect to the hh and vv axes; e.g., for the compression at corner e the equation would be

$$f=\frac{V}{A}-\frac{Vxb}{2I_v}+\frac{Vyd}{2I_h}.$$

Ordinarily if an eccentric load is used, it is applied in one of the principal axes, in which case the expression for f would include but two terms. If the applied force be a pull instead of a thrust, the same equation holds, but a positive result would give the tensile fibre stress.

The serious effect of an eccentric load may be readily determined by considering the cross-section of one of the columns designed in Art. 144. Suppose for example that the resultant force on the cross-section of the column shown in Fig. 224, instead of being applied at the centre of gravity be applied at a point two inches to the right of axis vv and two inches above axis cl. The compressive stress in the column will then evidently be a maximum at the corner marked x, and will be given by the following equation:

$$f=\frac{430{,}000}{33.47}+\frac{430{,}000\times2\times9}{1177}+\frac{430{,}000\times(2-1.30)\times(9.0-1.30)}{1531}$$
$$=20{,}900 \text{ lbs.}$$

If the load were to be applied at the centre of gravity, the corresponding fibre stress would be 12,800 lbs., hence the eccentricity produces an excess fibre stress of approximately 63%.

148. Effect of Combined Flexure and Thrust. While eccentricity of load increases materially the maximum stress on a column, it is frequently necessary or convenient to resort to this method of loading, such for example being the case with a building column supporting a crane-runway girder on a side bracket. Transverse flexure also occurs frequently; it is always present in a horizontal strut, such as a bridge chord, where its own weight may cause a considerable bending moment; it is also an important factor in the design of a top chord of a deck bridge when used to support the track ties directly. It is important, therefore, to be able to determine the maximum stress under such conditions.

The solution of the problem which follows, while not exact, is commonly used and gives a reasonable working method. The nomenclature refers to the cross-section of the column at the centre of the unsupported length, at which point maximum transverse deflection would occur.

Let M = initial bending moment at section due either to transverse loads or initial eccentricity.
M_1 = bending moment at section after column has deflected.
P = resultant force on section acting parallel to the column axis.
L = unsupported length of column.
δ = transverse deflection of column under load.
f = fibre stress in column due to bending moment.
s = maximum fibre stress in column.
c = a constant.
y = distance from centre of gravity of section to extreme fibre.

Now $M_1 = M + P\delta$; hence the solution of the problem requires the determination of δ. From the discussion of column formulas it is evident that δ cannot be accurately determined for columns of the lengths ordinarily used in practice, since it is partially due to variation in the initial condition of the column and to unintentional eccentricity of application of the load. For the case under consideration, however, assuming that the column is

stressed only to a reasonable working unit stress, the larger part of the deflection is due to the bending moment M. If it be assumed that the deflection of a column such as this may be expressed in terms of the fibre stress by formulas of the same general form as those for the deflection of a beam, the following equation may be written:

$$\delta = \frac{1}{c}\frac{fL^2}{Ey}.$$

$$\therefore \quad f = \left[M + \left(\frac{P}{c}\right)\left(\frac{fL^2}{Ey}\right)\right]\left(\frac{y}{I}\right),$$

whence

$$f = \frac{My}{I - \dfrac{PL^2}{cE}}.$$

$$\therefore \quad s = \frac{P}{A} + \frac{My}{I - \dfrac{PL^2}{cE}}.$$

For a beam supported at ends and loaded at centre $\frac{1}{c} = \frac{4}{48}$.

For a beam supported at ends and loaded uniformly $\frac{1}{c} = \frac{5}{48}$.

For a column it is common to assume $\frac{1}{c} = \frac{1}{10}$, giving the following equation in which the last term in the denominator is small and may often be ignored:

$$s = \frac{P}{A} + \frac{My}{I - \dfrac{PL^2}{10E}}. \qquad (28)$$

The actual fibre stress should not exceed that obtained by applying formula (22) or such other column formula as may be specified, being computed about the other axis.

A similar expression may be derived for a tension piece if the sign of the last term in the denominator be changed.

149. Building Columns under Eccentric Loads. The following discussion shows a method of computing the bending moment due to an eccentric load applied to a column of a one-story building by a traveling crane running on a track supported by brackets.

If the column be assumed as pin ended the curve of bending moments will be as shown in Fig. 230. The column is held at

the top by connection to the truss and at the bottom by friction at the base and foundation bolts, hence the bending moment, Px, of the eccentric load is resisted by horizontal forces at the ends of the column which form a couple the value of which is also Px. The maximum bending moment occurs at the load and depends upon the height at which the latter is placed. The maximum possible value is evidently Px, which would occur with the load at either end of the column. The curve of moments is represented as changing suddenly at the point of application of the load; this is not strictly correct, however, since such a condition could not actually occur if the load were applied to a bracket, as the latter would distribute its bending effect by means of the rivets connecting it to the column.

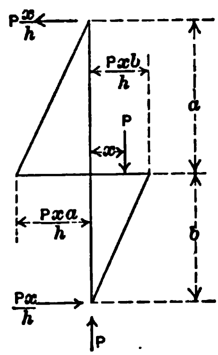


FIG. 230.—Curve of Bending Moments. Eccentrically Loaded Column.

It is seldom that such columns need to be treated as pin ended, since the ends are usually partially if not completely fixed. The effect of fixing the ends is to reduce the bending moments considerably.

Columns in high buildings are often eccentrically loaded and must be carefully studied. As such columns are usually continuous over a number of stories, and held more or less rigidly at each floor by the floor-beam connections or by wind bracing, this problem is a difficult one to treat mathematically and will not be considered at this point.

150. Design of Cast-iron Columns. The design of a cast-iron column differs somewhat from that of a steel column, hence the following treatment of hollow circular cast-iron columns under eccentric load is appended.

Mode of Procedure. 1st. Design the column for its direct load assuming a reasonable unit stress.

2d. Make the metal sufficiently thick to ensure a good casting. A thickness of 1 in. should, in general, be used, although in exceptional cases $\frac{3}{4}$-in. or thinner metal may be permitted.

3d. Compute the maximum fibre stress in the column at

designed in accordance with the foregoing requirements. In this computation any reasonable eccentricity of the load must be considered.

4th. If the fibre stress thus obtained differs too much from the allowable stress, revise the computation.

The following method illustrates the design of sucn a column, and shows a method of determining the eccentricity.

Assume beams A and B, Fig. 231, to each have a live reaction of 10,000 lbs., a dead reaction of 5000 lbs., and to be 12 ins. wide. Then the maximum load on the column = 80,000 lbs. Assuming a unit stress of 4000 lbs. for a trial section gives 20.0 sq.ins. A column 8 ins. in external diameter and of one inch material has an

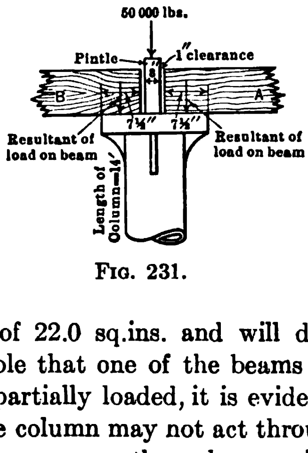


Fig. 231.

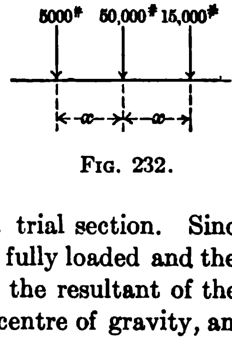


Fig. 232.

area of 22.0 sq.ins. and will do for a trial section. Since it is possible that one of the beams may be fully loaded and the other only partially loaded, it is evident that the resultant of the loads on the column may not act through its centre of gravity, and that in consequence the column will be eccentrically loaded. The maximum eccentricity will occur when one of the beams, say beam A, has its full live load, and the other is unloaded. The loading will then be as shown in Fig. 232, and the resultant will act at a distance from the line of action of the centre load equal to $x-\frac{60,000x}{70,000}=0.14x$, and will have a value of 70,000 lbs. The total load in this case is less than the maximum, but the effect of the eccentricity may be sufficient to make this the critical case.

In order to obtain the actual eccentricity it is necessary to ascertain the value of x. This involves the design of the column cap, and the location of the line of action of the resultant reaction on each beam. This latter cannot be ascertained with exactness,

since its distribution would depend upon the relative elasticity of the beam, column, and column cap, and upon the crushing strength of the wood. If the beam, cap and column were to be perfectly rigid, then the reaction on each beam would be distributed uniformly over its bearing surface; on the other hand if the column and cap were to be rigid and the beam elastic, the tendency would be to throw all the pressure to the edge of the cap, and to make that the point of application of the resultant. This latter condition could, however, not really be reached, since the wood would be crushed at the point of bearing, which would relieve the pressure there and distribute it over a greater length of beam. The true position of the resultant is evidently somewhere between the centre of bearing and the edge of the cap.

To design the cap and determine the position of the resultant reaction let the following assumptions be made:

1st. Pressure varies uniformly from a maximum at edge of cap to zero at end of beam.

2d. Pressure at edge of cap under maximum load equals allowable crushing strength of the wood across the grain, which may be assumed as 350 lbs. per sq.in. for yellow pine beams, or 4200 lbs. per lineal inch for a 12-in. beam. Fig. 233 shows the distribution of pressure at the end of one of the beams based upon the assumption just made. The distance d may be determined by dividing the maximum beam reaction by one-half the allowable crushing strength per lineal inch. For the case under consideration, this gives $7\frac{1}{2}''$ approximately.

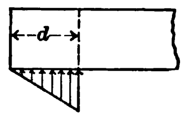


FIG. 233.

The value of x is

$$1\tfrac{1}{2}'' + 1'' + (\tfrac{2}{3})(7\tfrac{1}{2}'') = 7.5''.$$

The eccentricity under the partial loading is then $(7.5)(0.14) = 1.05$ ins. The eccentricity due to bending of the column will be neglected here, as being an unnecessary refinement for a material as variable as cast iron, hence the fibre stress due to the eccentricity will be

$$\frac{4 \times 70{,}000 \times 1.05}{.049(8^4 - 6^4)} = 2140 \text{ lbs.}$$

To determine whether the column is safe this eccentric stress should be added to the maximum stress due to the direct load as determined by the column formula, and the sum should not exceed the allowable unit stress in a short column. If formula (27) be applied, the maximum fibre stress due to direct load is given by the expression

$$f=\frac{P}{A}+\frac{32L}{d}=\frac{70,000}{22}+\frac{32\times 14\times 12}{8}=3180+670=3850.$$

The eccentric stress added to this gives a total of 5990 lbs., which is less than 6100 lbs., the allowable unit stress by the formula for short columns, hence the column has an area that is only slightly in excess of the required amount and may be used.

151. Design of Iron and Steel Tension Members. The design of tension members involves little more than the selection of bars with sufficient net area to carry the total stress without exceeding the allowable unit stress. Steel or iron tension members may be divided into two general types: viz., solid bars rectangular or circular in cross-section, and built-up members composed of structural shapes riveted together. Solid bars are used generally in pin trusses for diagonals and bottom chord members, and in Howe trusses for verticals. Built-up members are generally employed for tension members in riveted trusses and for the end hangers in pin trusses.

Of the first type of member the eye bar shown in Fig. 234 is used most commonly. Such bars are made by most of the large

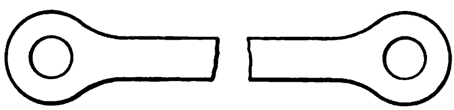


FIG. 234.

steel manufacturers and are fully described in their handbooks. The heads of these bars are designed so that the bar if tested to destruction, will fail in the body rather than in the head, and the engineer should specify that full-sized tests should give this result and not attempt to proportion the heads. In determining clearance, the dimensions of the heads given by the makers may be used, noting that the diameter of the head de-

pends upon the size of pin hole. Eye-bars may be manufactured to any thickness above the minimum size quoted by the makers, but a thickness above 2 ins. should not generally be employed, since such thick bars are not likely to be of the best material. A good rule to observe in selecting bars is to keep the thickness between one-sixth and one-third the width. Eye-bars are generally used in pairs, since an odd number of bars would give a

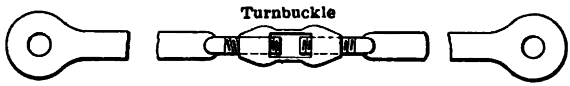


FIG. 235.

poor arrangement on the pin. For counters, adjustable eye-bars such as those shown in Fig. 235 may be used, the two bars being connected by a turnbuckle or sleeve nut; iron rods with loops formed by welding such as those shown in Fig. 236 may be

FIG. 236.

used if the counter stresses are small. For the verticals of Howe trusses iron rods, with screw ends fastened by nuts bearing on washers supported by the top chord, are generally employed.

In proportioning adjustable members allowance must be made

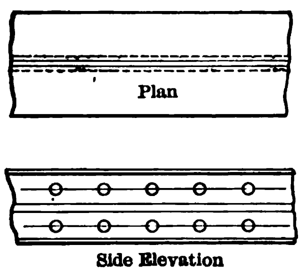


FIG. 237.

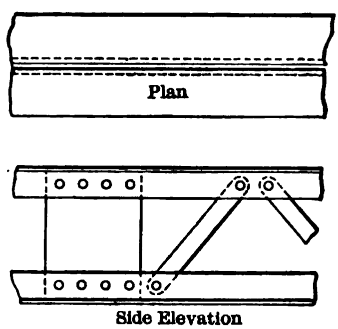


FIG. 238.

for the decrease in section due to the screw threads. It is usually advisable to upset the screw end, that is, to make it of larger diameter than the body of the bar, so as to give sufficient area at

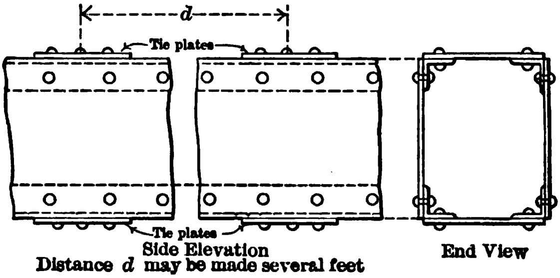


FIG. 239.

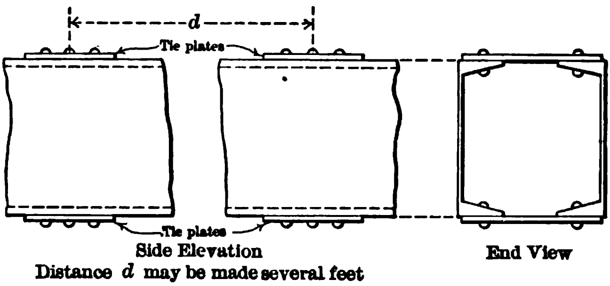


FIG. 240.

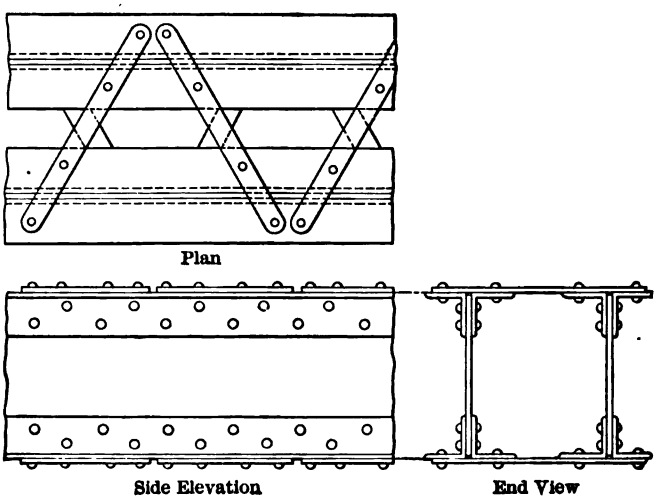


FIG. 241.

the root of the thread to make the bar as strong there as elsewhere. For short rods, however, the labor cost involved in this process may be greater than the saving of material would warrant.

Riveted tension pieces may be made of various sections. Figs. 237 to 241 inclusive show typical members, and need no explanation. While these members do not need latticing or tie plates to keep the separate parts from buckling, some connection between them should be used to make the different parts act together. The design of these details must be left to the judgment of the engineer.

CHAPTER XII

PIN AND RIVETED TRUSS JOINTS

152. Bridge Pins Described. A bridge pin may be considered as a large rivet which has to carry bending moment as well as shear and bearing. The difference between a bridge pin and a rivet is due to construction. A rivet is driven while red hot, and is then headed, usually under a high pressure, so that it completely fills its hole and binds together so tightly the different pieces through which it passes that there is little, if any, opportunity for it to become distorted through bending. A bridge pin, on the other hand, is always made somewhat smaller than the pin hole, and the attempt is not made to hold together tightly by the pin the members coming on it, hence it can bend and must be designed to resist bending moment as well as shear. It must also have sufficient bearing area on each connected piece to make it safe against failure by crushing of either pin or member, this latter frequently being secured by increasing the thickness of the member by the addition of a plate or plates rather than by an increase in the diameter of the pin.

153. Arrangement of Members on Pin. The actual design of a pin as carried out in practice is a very simple process after the arrangement of the different members upon the pin is once satisfactorily accomplished. To properly arrange the members is, however, a somewhat complicated problem, since the arrangement on one pin cannot be worked out independently, but must be studied with due regard to its effect upon the other parts of the truss.

The following rules should be observed in arranging the different members:

1st. Allow sufficient clearance. This is extremely important, since insufficient clearance gives trouble in erection. For trusses of ordinary spans, the heads of all eye-bars coming on the pin should be assumed as $\frac{1}{16}$ in. thicker than their normal thickness

and the total clearance between riveted members should be at least one-half inch. Fig. 242 shows the method of providing clearance in a simple case.

Distance "a" = distance between chord webs, should, for case shown, be made

$$13\tfrac{1}{4}+2(1\tfrac{1}{2}'')+2(\tfrac{5}{8})+\tfrac{1}{2}=18''.$$

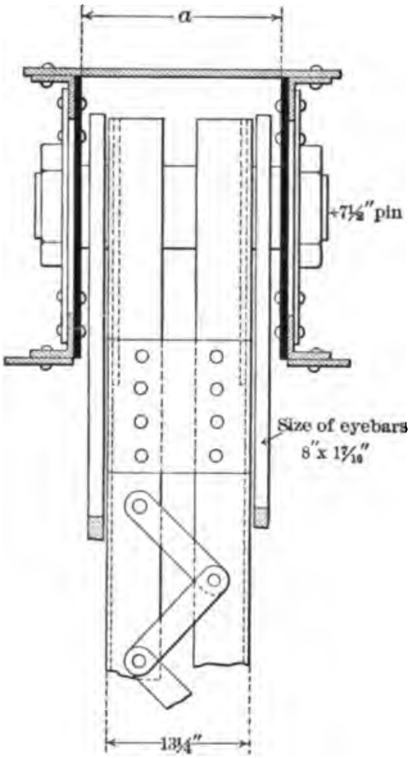


FIG. 242.

If rivets are countersunk, this distance may be reduced by 1¼ in. If rivet heads are flattened or are countersunk but not chipped the distance between channels may be varied accordingly. (Note that rivet heads if countersunk but not chipped usually project ⅛ in. above the surface and that rivet heads are frequently flattened without being countersunk, so that they project but ⅜ in. above the surface.)

2d. Arrange eye bars so that their centre lines will be parallel or nearly so with the centre line of the truss. It is seldom possible if compact joints and small pins are to be obtained to follow this rule very closely, but it is common to specify that no bar shall deviate from the centre line of the truss by more than $\frac{1}{16}$ in. per foot in length of the bar. In cases where a greater allowance than this is necessary the bar should be bent to the proper slope before being annealed.

As it is sometimes difficult to arrange the different members so that all the above conditions will be observed, the student is advised to lay out to a large scale, say $1\frac{1}{2}$ in. = 1 ft., the different joints of each chord. The distance apart of the various joints should also be plotted to scale, but this scale may be much smaller than the scale of details. The different members can then be drawn from joint to joint and the deviation from the centre line can be determined by scale. This method is indicated in Fig. 243 for two joints of a bottom chord. To carry out the method

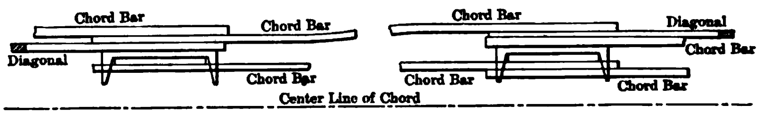


FIG. 243.

completely the chord should be drawn on a sheet of sufficient length to show all the joints, or if the truss is symmetrical, all the joints up to and including the centre joint, and the top chord should be plotted above the bottom chord in a similar manner.

It will be noticed that, in order to secure the above arrangement, the channel flanges may have to be cut away. This is undesirable, but is frequently necessary in order to avoid the necessity of using a pin of large diameter. If this is done, the channels should be reinforced by web plates extending throughout the distance over which the flanges are cut, unless the channel webs alone without the flanges are of sufficient strength to carry the compression. In investigating this case in a compression member, the column formula should be applied, using for the unsupported length the distance from the centre of pin hole to the first row of rivets beyond the point where the flanges are cut.

154. Minimum Size of Pins. Before it is possible to complete the arrangement of the various members, it is necessary to make some assumption as to the size of the pins, since it is usually necessary to add pin plates to the riveted members either for the purpose of strengthening the member against crushing on the pin, or to make up for the area taken out by the pin hole, since the pin, unlike a rivet, does not completely fill the hole and hence cannot be counted upon to carry compression. To determine the size approximately requires some experience; the lowest limit is, however, usually fixed by the width of the widest eye-bar connected to the pin as shown below.

Let f_b = allowable bearing stress per square inch on pin.
f_t = allowable tension stress per square inch in bar.
w = width of widest bar coming on the pin.
t = thickness of same bar.
d = diameter of the pin.

Then

$$f_b dt = \text{bearing value of the bar on the pin,}$$

and

$$f_t wt = \text{tensile strength of the bar.}$$

Putting these equal gives

$$f_b dt = f_t wt.$$

Therefore,

$$d \geq w\frac{f_t}{f_b}.$$

For example if $f_t = 16{,}000$, $f_b = 24{,}000$, and the width of the widest bar coming on the pin is 6 in., the diameter of the pin should not be less than $\frac{16}{24} \times 6 = 4.0$ in., hence the pin in this case should be assumed as not less than 4.0 in. in diameter. Whether it should be assumed as larger is a matter which can only be estimated by experience, but it should be noted that it is wiser to assume the pin *too small* rather than too large, since, in the former case, pin plates, which are somewhat thicker than are needed, will be selected at first and these may be easily reduced in thickness if it be found that the diameter of the pin should be larger than that assumed. The only exception to the statement "that it is usually on the safe side to assume the pin too small"

is when a reinforcing plate is not needed to increase the bearing resistance on the pin, but is required to make up for reduction in section by the pin hole. This sometimes happens near the centre of the top chord of a simple truss, but in arranging the members on a pin it is wise to always allow for at least one pin plate upon the chord at every joint; a $\frac{3}{8}$-in. plate if rivets do not require countersinking, and a $\frac{7}{16}$-in. plate if the clearance is so small as to make countersinking necessary.

155. Stresses Causing Maximum Moment and Shear. After the arrangement of the members is satisfactorily accomplished it is necessary to compute the maximum stresses which act simultaneously on each pin, and which seem likely to produce critical bending moments and shears.

In order to obtain these simultaneous stresses it is sometimes necessary to calculate anew the stresses in a number of bars under the loading which produces the maximum in one of them. For example, for the truss shown in Fig. 244, the maximum

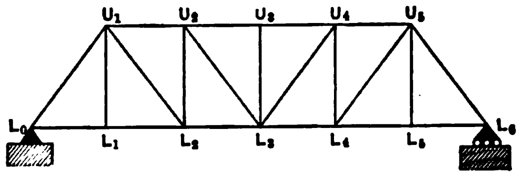


FIG. 244.

moment on pin L_2 may occur under the loading which produces maximum stress in chord L_1L_2, diagonal U_1L_2, or chord L_2L_3, hence it becomes necessary to compute the stress in the bars connected by pin L_2 under all of these conditions of loading.

In all cases it is the horizontal and vertical components of the stresses which are desired, and the results should be checked by noting whether the pin is in equilibrium under the action of these components, that is, whether $\Sigma H = 0$ and $\Sigma V = 0$. There is one point here which may cause trouble. The floor beam in an ordinary bridge is frequently connected to the post above the pin, hence the post stress which reaches the pin is not the stress in the post as a whole, but is the stress in the post below the floor beam. To avoid confusion, no attention should be paid to the actual stress in the post, whether the floor beam is

above or below the pin, but the stress coming to the pin from the post should be placed equal to the vertical component of the diagonal stress. After the stresses are found, it is desirable to make sketches for each joint showing the stresses in the bars meeting at the joint.

156. Computation of Maximum Moment and Shear. The next step is to determine the maximum bending moment and shear on the pin for each loading. This can best be accomplished for the moment by plotting the curves of vertical and horizontal moments, and determining by inspection or trial the section where the maximum resultant moment occurs. This resultant can be determined with sufficient accuracy by graphical methods, since its value equals that of the hypothenuse of a right-angled triangle, the sides of which equal the vertical and horizontal moments respectively.

It is seldom that the maximum shear needs to be carefully figured, since ordinarily the bending moment determines the size of the pin. The shearing stress should always be investigated, however, and in doubtful cases its maximum value determined by the method given above for bending moment.

If the size of the pin as computed differs materially from that assumed, the thickness of the pin plates should be investigated and revised if necessary. This should not be done too hastily, however, since it is customary to use but few different sized pins, in a truss, and it may happen that the pin as computed may not be the one which it is finally decided to use. Examples of pin computation will now be given.

157. Computation of a Top Chord Pin for Truss Shown in Fig. 245.

Problem. Determine the size of pin and thickness of bearing area of chord and vertical at joint U_2, using following allowable unit stresses:

Bearing on pin, 22,000 lbs. per square inch,
Bending on pin, 22,500 lbs. per square inch,
Shear on cross-section, 10,000 lbs. per square inch.

Solution. For this pin the only loading which needs to be considered is that which produces the maximum stress in diagonal U_2L_3. The reason for this is that the top chord is continuous at the joint, and spliced elsewhere as shown. This is inconsistent with the theory upon which the computation of truss stresses is based, but is the common practice and probably does not affect the stresses materially, while it simplifies

greatly the construction. Under this condition the duty of this pin is to connect the diagonal to the top chord and vertical, the horizontal

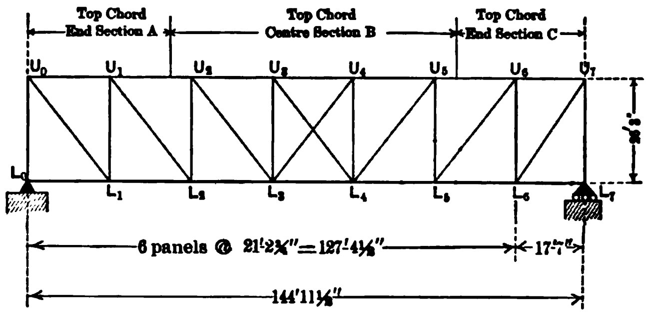


FIG. 245.

(For composition of members, see opposite page.)

component of the diagonal stress being transmitted by the pin to the chord, and the vertical component to the post. The actual stress in the chord, therefore, is not an element in the pin design, and needs to be considered only in investigating the strength of the chord at the cross-section through the pin hole. Fig. 246 shows the maximum stress in the diagonal with its vertical and horizontal components.

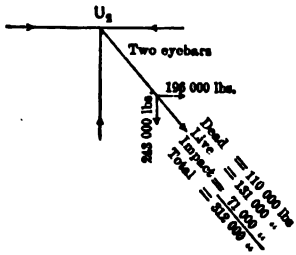


FIG. 246.

The allowable unit stresses of 16,000 lbs. per square inch tension, and 22,000 lbs. per square inch bearing give for the minimum size pin required for bearing on the 6-in. diagonal bar, $\frac{16}{22}\times 6'' = 4.35''$.

Since the stress in the diagonal is large the size of pin which will be assumed in determining bearing areas will be taken as somewhat larger than the minimum size, or say 5¼ in. The bearing area required by this assumption may then be computed, by assuming the stress to be distributed uniformly over a surface equal to the plane diametrical section of the pin.

Total thickness of bearing required on chord

$$= \frac{196{,}000}{22{,}000 \times 5\frac{1}{4}''} = 1.70''.$$

Total thickness of bearing required on vertical

$$= \frac{243{,}000}{22{,}000 \times 5\frac{1}{4}''} = 2.10''.$$

SCHEDULE OF SIZES FOR TRUSS SHOWN IN FIG. 245.

Top chord Section A.	One cover plate Two top angles Two bottom angles Two webs	28"×½" 4"×4"×½" 5"×3½"×½" 22"×½"
Top chord Section B.	One cover plate Two top angles Two bottom angles Two webs Two webs	28"×½" 4'''×4"×7/16" 5'×3½"×7/16" 22"×7/16" 14½"×7/16"
Top chord Section C.	One cover plate Two top angles Two bottom angles Two webs	28"×½" 4"×4"×⅜" 5"×3½"×⅜" 22"×½"
L_0L_1 and L_6L_7 L_1L_2 L_2L_3 L_3L_4 L_4L_5 L_5L_6	Two channels Two eye bars Four eye bars Four eye bars Four eye bars Two eye bars	12"−25 lbs. 7"×2" 7"×1 11/16" 7"×1 13/16" 7"×1 9/16" 7"×1 11/16"
U_0L_0	One cover plate Two webs Two webs Four angles	21"×7/16" 20"×½" 13"×9/16" 5"×3½"×9/16"

U_1L_1	Two channels Two plates	15"−40 lbs. 12"×½"
U_2L_2, U_3L_3, U_4L_4	Two channels	15"−33 lbs.
U_5L_5	Two channels	15"−40 lbs.
U_6L_6	Two channels Two plates	15"−40 lbs. 12"×9/16"
U_7L_7	One cover plate Two webs Two webs Four angles	21"×7/16" 20"×½' 13"×9/16" 5"×3½"×9/16"
U_0L_1	Two eye bars Two eye bars	7"×1 11/16" 7"×1 9/16"
U_1L_2	Two eye bars Two eye bars	7"×1" 7"×1¼"
U_2L_3 U_3L_4 U_6L_3 U_5L_4	Two eye bars Two eye bars Two eye bars Two eye bars	6"×1 9/16" 5"×⅞" 5"×1" 6"×1¾"
U_6L_5	Two eye bars Two eye bars	7"×1" 7"×1 7/16"
U_7L_6	Two eye bars Two eye bars	7"×1 7/16" 7"×1 11/16"

In order to obtain these thicknesses it is necessary to add a $\frac{11}{16}$ in. pin plate to each half of the vertical. The chord thickness need not be increased for bearing, but a plate should be added to make up for the reduction in area due to the pin hole. This reduction

$$=5\tfrac{1}{4}\times1\tfrac{3}{4}=9.2 \text{ sq.ins.}$$

A $22''\times\frac{7}{16}''$ pin plate on each rib of the chord gives a net area of $16\frac{3}{4}\times\frac{7}{8}=14.7$ sq.ins., which is ample. A thinner plate should not be assumed, since the rivets may have to be countersunk, and it is inadvisable to countersink $\frac{7}{8}$-in. rivets in a plate thinner than $\frac{7}{16}$ in.

The width adopted for the top chord is $18\frac{1}{2}$ in. between the 22-in. webs, and for the vertical 12 in. out to out of webs. These widths are determined, principally, by the conditions at the end joint, which will not be considered here. It should be noted, however, that the width of the vertical is given for the distance out to out of webs, instead of between webs, the flanges being turned away from each other. This distance is made constant for all verticals, so that the lengths of the floor beams may be the same regardless of the thickness of the post channel webs. The arrangement of members adopted at the joint is shown by Fig. 247.

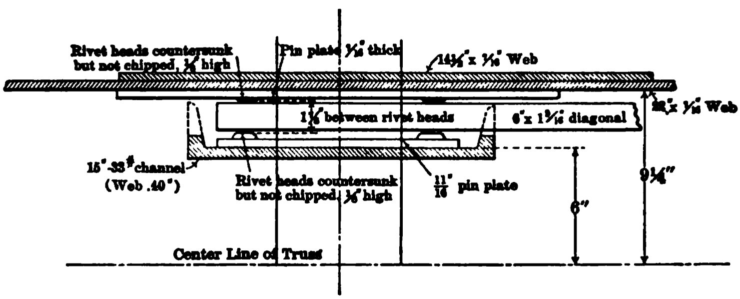


FIG. 247.

The forces acting on the pin were assumed, in determining the required bearing area, as distributed uniformly over a plane surface equal to the diametrical section of the pin. In computing moments, however, it is the usual custom to consider these forces as concentrated at the centre of the bearing areas. The distance between the points of application of these stresses should be computed upon this basis, and the bending moment and shear on the pin determined. The results of these computations are shown in Fig. 248.

It is evident that the maximum moment in this case is the resultant of the maximum horizontal and the maximum vertical moments, since these both occur at the same section. This is found graphically, as shown by Fig. 249, and equals 262,000 in. lbs.

With an allowable fibre stress in bending in the pin equal to 22,500 lbs., a 5-in. pin is required to carry this moment. (See table for

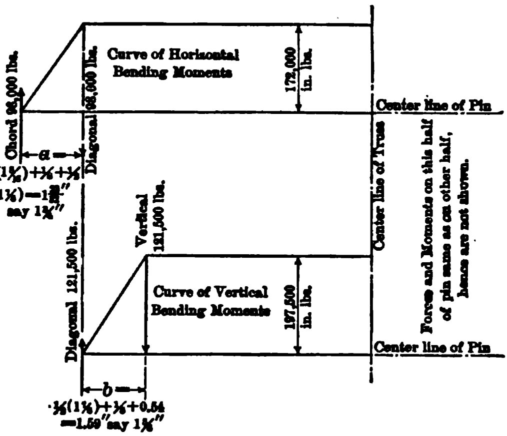


FIG. 248.

"Maximum Bending Moments on Pins" in "Cambria Steel," 1914 edition, page 332). This pin may be used, provided it is strong enough to carry the maximum shear. (The pin plates used on post and chord are both somewhat in excess of the size needed for a $5\frac{1}{2}$-in. pin, and hence need not be recomputed.) The area of a 5 in. pin is 19.6 sq.ins. which at a unit stress of 10,000 lbs. per square inch assumed to be uniformly distributed over the cross-section, is good for 196,000 lbs. shear. The maximum shear is 121,500 lbs., which is less than the allowable shear, hence the 5-in. pin satisfies all the necessary requirements and should be used.

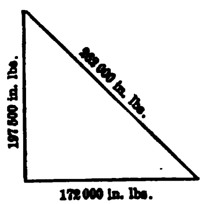


FIG. 249.

158. Computation of a Bottom Chord Pin for Truss Shown by Fig. 245.

Problem. Determine the size of pin and thickness of bearing area on vertical at joint L_2, using same unit stresses as for pin U_2.

Solution. For this pin two loadings must be considered.

1st. That which produces maximum stress in chords L_1L_2 and L_2L_3.

2d. That producing maximum stress in diagonal U_1L_2.

For the first case, the chord stresses are identical with the maximum stresses, since the chords of this truss were computed for a uniform load

per foot, and hence these stresses may be written down at once. The difference between the chord stresses equals the horizontal component of the diagonal stress, from which the vertical component is readily obtained. The stress transferred to the vertical from the pin equals the vertical component of the diagonal stress. The stresses for this case are shown by Fig. 250.

For the second case it is necessary to compute the stresses in the chords produced by the loading which causes maximum stress in the diagonal. This computation requires but little additional work, even if a concentrated load system is used, since the position of the loads is known and the left reaction would have been determined in making the shear computations. The stresses for this case are shown by Fig. 251.

Had the maximum chord stresses been computed for a concentrated load system it might have been necessary to compute the pin for three instead of two cases, since the position of the loads for maximum stress in chord L_1L_2 might have been different from the position for maximum stress in the chord L_2L_3. It should be noted, however, that in pin com-

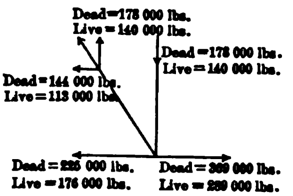


FIG. 250.

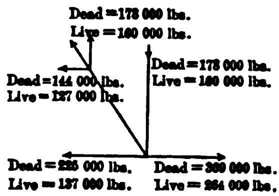


FIG. 251.

putations a uniform load giving the same maximum stress as that occurring in any one of the bars connected by the pin may be used if desired in determining the simultaneous stress in the other bars, the error being slight. It will be noticed that impact is not included with above stresses. The reason for this is that the allowance for impact if figured by the formula $I = S\left(\frac{S}{S+D}\right)$, which will be used for this case, would give different percentages for the different bars, with the result that the forces on the pin would not balance. It is necessary, therefore, in such a case to compute the dead and live moments separately, and determine the impact as a factor of the moments and not of the bar stresses. If the impact is computed on the basis of the percentages of loaded length, it might if desired be included in the stresses, since the loaded length is the same for each case for all the bars concerned.

The minimum size of pin for this joint is $\frac{16}{22} \times 7'' = 5.1''$. The stresses are large, hence it would seem reasonable to assume a pin somewhat larger than this, and a 6-in. pin will be taken. For the loading of the second case, the post stress, with impact added, equals 414,000 lbs.; hence the

total thickness required for bearing on the 6-in. pin is $\dfrac{414{,}000}{22{,}000 \times 6} = 3.14$ in. The thickness of the channel web is 0.40 in., hence to each web must be

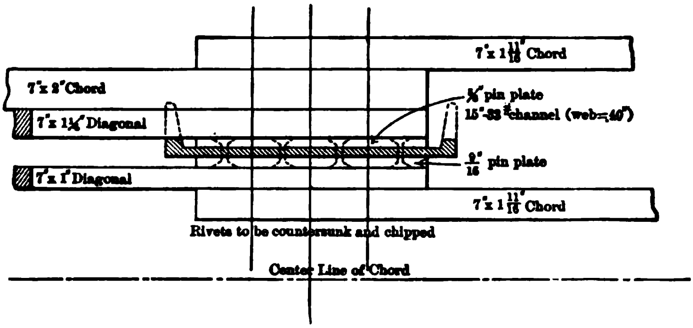


FIG. 252.

added 1.17-in. pin plates or say one $\frac{9}{16}$-in. plate and one $\frac{5}{8}$-in. plate. The proposed arrangement of the different members coming on the pin is

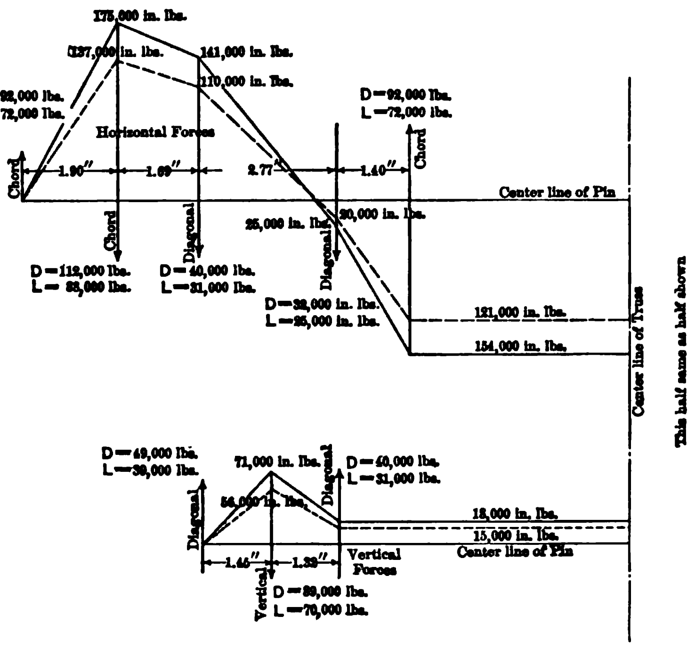


FIG. 253.—Curves of Moments for Case 1. Full Lines are Dead Moments.

shown in Fig. 252. This arrangement is one which gives a satisfactory location of the bars as regards the other joints of the truss.

Figs. 253 and 254 show curves of moments for both loadings, and should be understood without difficulty. It will be noted that in determining distances between loads each eye-bar is assumed to be $\frac{1}{16}$-in. thicker than its nominal size.

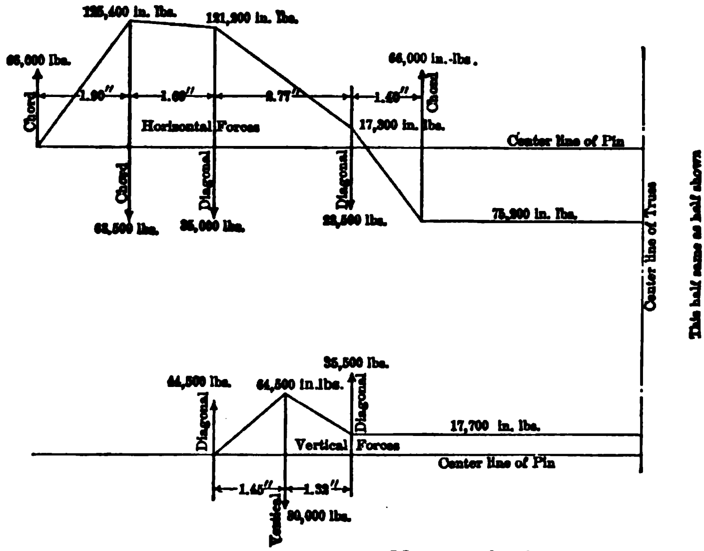


FIG. 254.—Curve of Live Moments for Case 2.

It is evident that the maximum moment on the pin occurs for Case 1 and equals:

$$
\begin{aligned}
D &= 175{,}000 \text{ in.lbs.,}\\
L &= 137{,}000 \text{ in.lbs.,}\\
I &= 60{,}000 \text{ in-lbs. (by formula (8))}.\\
\hline
& 372{,}000 \text{ in.lbs.}
\end{aligned}
$$

The size of pin required to carry this moment with unit stress of 22,500 lbs. is 5$\frac{5}{8}$-in (see Cambria Handbook). This is somewhat smaller than the size assumed in computing the thickness of the bearing plates on the post. As the thickness of these bearing plates has no influence upon the maximum moment on this pin which occurs at the next to the outermost chord bar, it is evident that the 5$\frac{5}{8}$-in. pin may be used without recomputation by making one of the pin plates on the vertical somewhat thicker than the size required for a 6-in. pin, and that no other change is necessary.

Shear. The allowable shear on the 5$\frac{5}{8}$-in. pin at 10,000 lbs. per square inch equals 248,500 lbs.

A slight study of the pin and its applied loads shows that the maximum shear for Case 1 is:

$$
\begin{aligned}
D &= 92{,}000 \text{ lbs.} \\
L &= 72{,}000 \text{ ''} \\
I &= 32{,}000 \text{ ''} \\
\hline
& \;\;196{,}000 \text{ ''}
\end{aligned}
$$

For Case 2, the shear is still less, hence the $5\frac{5}{8}$-in. pin is strong enough to carry the shear.

159. Effect Upon Pin of Change in Arrangement of Members. The student should consider carefully the comparatively great effect upon the moment of a slight change in the arrangement of the members on pin L_2. If the 2-in chord bar were to be interchanged in position with the $1\frac{1}{4}$-in. diagonal, the maximum dead moment due to horizontal forces would be increased by 54,000 in.lbs. and

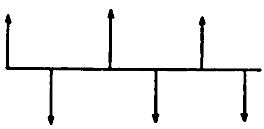


FIG. 255.

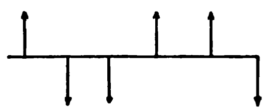


FIG. 256.

the live moment proportionally. The effect of an interchange of the 2-in. chord bar with the adjoining $1\frac{1}{16}$-in. bar would be to increase the horizontal dead moment by 363,000 in.lbs. and the live moment proportionally. It would also make the maximum horizontal moment occur at a section where vertical moment would exist, hence the maximum would be a resultant of horizontal and vertical moments instead of a single moment, as is now the case, and this would still further increase the moment.

It is desirable to use as small pins as possible, so that the size of the eye-bar heads may be kept within reasonable limits, hence the arrangement of the bars should be carefully studied, and the designer should bear in mind that an arrangement which will produce both positive and negative moments will usually give a satisfactory result. For example, if the arrangement of bars shown in Fig. 255 be changed to correspond to that shown

in Fig. 256, the moment will be reduced, since in the first case the moment continually increases while in the second case the moment varies from positive to negative and then back to positive, its maximum value being far below that reached in the first arrangement.

The thickness of the bars also has an important effect upon the size of the pin, and a reduction can often be made by reducing in thickness one bar of a member and increasing another by the same amount. This, of course, cannot be done if the member is composed of only two bars, since in such a case both bars must be equal in size to preserve the symmetry of the truss.

160. Pin Plate Rivets. The determination of the number of rivets required in the pin plates sometimes requires careful study. The student should, however, have no difficulty in solving this problem if he is careful to use enough rivets to carry from each plate the stress which it receives from the pin, assuming that it receives that proportion of the total stress which its thickness bears to the total thickness of bearing. Due allowance should be made in case several pin plates are needed for the effect of intermediate plates upon the strength of the rivets, and it is often found desirable to make plates of different lengths so that something of the effect of a tight filler may be obtained.

161. Pin Nuts. The nut commonly used on bridge pins is a special nut which is much thinner than the ordinary hexagonal or square nut, since its function is not to carry tension into the pin, but merely to hold the bars in place. It should be held in position by a cotter pin, since nuts not held have been known to be loosened by the impact of trains, and to fall off. On very large trusses, nuts are sometimes replaced by washers which are held in place by a rod passing through a hole bored along the longitudinal axis of the pin.

162. Packing Rings. In order that the bending moment on the pin may not differ from the computed value, it is necessary that the eye-bars be held in the position assumed in the computations. To do this, it is necessary to use washers or collars between some of the bars. These are sometimes made of thin plates bent around the pin, and sometimes of short pieces of iron pipe. When the bar is restrained by the other members so that the clearance is not more than $\frac{1}{4}$ in. to $\frac{1}{2}$ in. the use of such washers is unnecessary.

163. Riveted Truss Joints. The design of the joints of riveted trusses is of equal importance with the design of the main members and should receive most careful study. The observance of the following rules is necessary in order to prevent eccentricity in the application of the forces to the members meeting at a joint and consequent increase in fibre stress in the main members.

1. Centre of gravity lines of members meeting at a joint should intersect at a point.

2. Connection rivets in a member should be arranged symmetrically about the axis passing through its centre of gravity, with as few rivets as practicable in a line parallel to its longitudinal axis.

3. Members composed of a single angle, or of two angles back to back, should be connected to plates by means of lug angles in the manner illustrated by Fig. 257, or else the allowable unit stress in the member should be reduced to provide for the eccentric application of the load. The use of the lug angle is often desirable, not only to prevent the eccentricity of application of the load, but also to decrease the size of the connection plate which would otherwise be necessary.

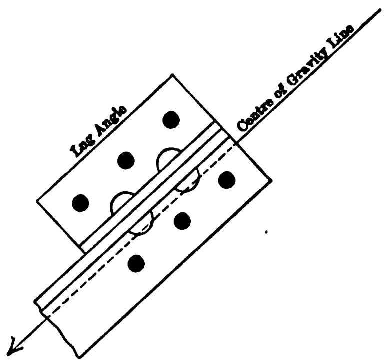


FIG. 257.

4. If stress at any joint is to be transferred from one member into a gusset plate and thence transferred to another member, the group of connection rivets in the second member should have its centre of gravity coincident as nearly as possible with the point of intersection of the two members.

5. The arrangement of the connection rivets in a tension member should be such as to reduce the cross-section area of the member as little as possible consistent with economy in the connection plate. In order that this result may be obtained

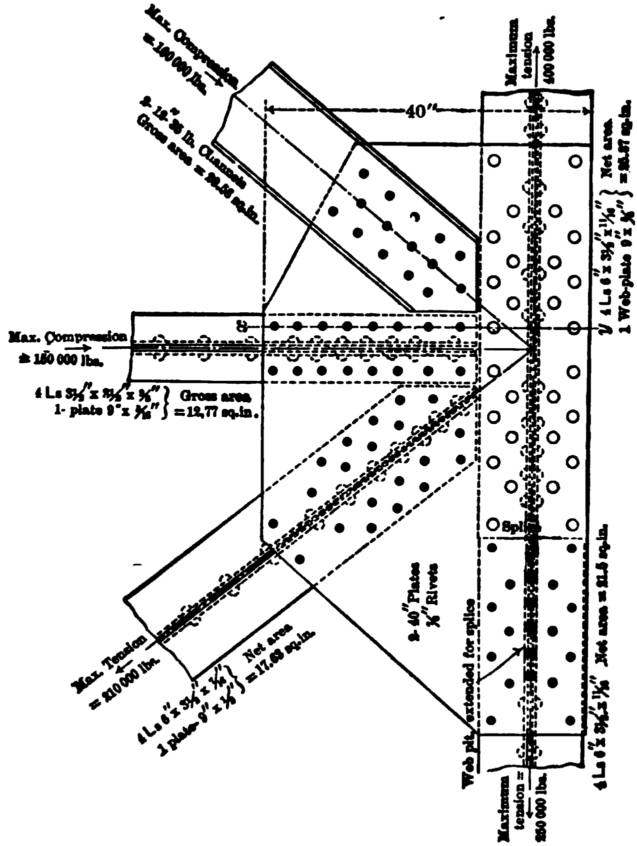


FIG. 258.

it is usually desirable to have not more than two rivets at right angles to the line of action of stress in the row farthest from the point of intersection of the members meeting at the joint, and to make the distance between this row and the row next to it as much as 5 ins. in order that rivets in outstanding legs of angles or in channel flanges may stagger completely with the connection rivets. It should be observed that the connection rivets gradually trans-

fer the stress from a member to a plate, and that in consequence the required net area of the member decreases in passing from the edge of the plate toward the end of the member; hence as the latter point is approached, the reduction in area of the member due to rivet holes may be very large without reducing the strength of the member.

6. The size and thickness of connection plates should be determined by the following considerations:

(*a*) The size of connection plates must be sufficient to enable the rivets necessary for connecting the different members to be properly located. In general it is desirable to use a small rivet pitch; usually for $\frac{7}{8}$-in. rivets a 3-in. pitch, except where a larger pitch is required by the application of rule 5.

(*b*) The net section across the plate at right angles to the line of action of a member must be sufficient to carry that proportionate part of the stress in the member which is transferred to the plate by the rivets between the given section and the end of the plate.

(*c*) If the resultant stress upon any section of a connection plate is eccentrically applied, as determined by assuming each rivet on one side of the section to carry to the plate its proportionate part of the total stress in the connecting member, the plate must be made of sufficient thickness to withstand the effect both of this eccentricity and the direct stress upon the section.

Fig. 258 shows a typical joint in a riveted truss and illustrates the application of some of these principles. It also shows a splice in a tension member in which the connection plate is used as a splice plate. This is a common practice, and whether the splice be of a tension member or a compression member, sufficient rivets should be used in the splice plates to carry the entire stress, no dependence being placed upon the abutting of the ends of the members.

The following example illustrates the character of the computations necessary to determine the thickness of such a plate:

Problem. Determine the necessary thickness of the connection plate shown in Fig. 258, using an allowable unit stress in bending of 16,000 lbs. per square inch.

Solution. Inspection of the plate indicates that section xy is probably the critical section, since it contains many rivet holes, and the resultant stress on either side of it is large in magnitude and applied at some dis-

tance below the centre of gravity of the cross-section. The strength of the plate at this section will therefore be investigated.

The forces acting to the right of xy are the proportionate part of the chord stress carried into the plate by the chord rivets, and the total stress in the diagonal. Evidently the stress due to the chord is the more important factor, since its line of action is further from the centre of the cross-section, hence the condition of loading corresponding to maximum stress in this chord bar will be assumed. Computations show that for this condition the stress in the diagonal is 100,000 lbs. The stress passing into the plate from the chord rivets will be taken as the product of the number of rivets to the right of xy and the allowable stress per rivet, it being assumed that the total number of rivets is little if any in excess of the number actually needed. The assumption that the thickness of the plate will be such as to cause the rivets to be limited by shear rather than bearing and that the allowable unit stress in shear is 12,000 lbs. per square inch, gives a total force of $14 \times 7200 = 100{,}800$ lbs., the rivets cut by the section being included, since they bear upon the portion of the plate to the right of xy rather than to the left.

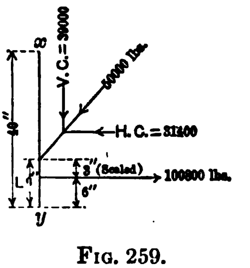


FIG. 259.

The forces acting upon section xy of one of the two connection plates will, therefore, be as shown in Fig. 259.

It has already been shown in considering plate girder web splices that the effect of a row of rivet holes such as exist at section xy in reducing the strength in bending will probably be amply allowed for if the moment of inertia be considered as three-quarters of the value for the gross cross-section. If this allowance be made the maximum stress in the plate, assuming its thickness as t will be given by the following expression:

$$f=\frac{100800-31400}{(40-10)t}+\frac{4}{3}\cdot\frac{6(100800\times14-31400\times11)}{t(40)^2}$$

$$=\frac{69400}{30t}+\frac{106580}{20t}=\frac{7642}{t}.$$

Since the allowable value of f is 16000 lbs. $t=\frac{7642}{16000}=\frac{1}{2}''=$ required thickness. This thickness would develop more than the shearing value of the rivets, and is consequently sufficient, at least for the section investigated.

A more accurate determination could be made if thought desirable by actually determining the net moment of inertia, and other sections may be tested in a similar manner if doubt exists as to the critical section.

CHAPTER XIII

GRAPHICAL STATICS

164. Graphical and Analytical Methods Compared. It is generally possible to solve by graphical methods all statical problems which can be solved analytically, while for certain classes of problems such methods are somewhat simpler and more rapid than analytical methods, such, for example, being the case in the problems of Arts. 89 and 90. As a general rule, however, analytical methods are more satisfactory both in accuracy and speed. The engineer should nevertheless be thoroughly familiar with the principles of graphical statics so that he may be prepared to apply them, particularly in checking analytical computations. A knowledge of them is also necessary in order that engineering literature may be read intelligently. For a comprehensive treatment of the subject the student is referred to "Graphische Statik," by Muller-Breslau.

165. Force and Funicular Polygons. The most obvious method of determining graphically the magnitude, direction, and point of application of the resultant of a set of coplanar forces may be briefly stated as follows:

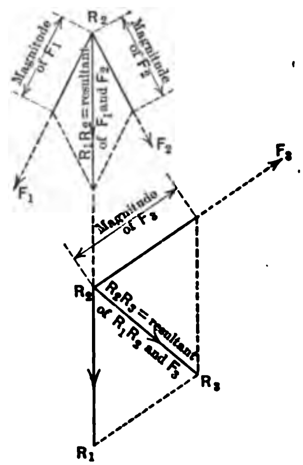


FIG. 260.

Plot the correct position and direction of the forces as indicated in Fig. 260 by F_1, F_2, and F_3. Prolong any two forces, such as F_1 and F_2, until they meet, thus obtaining the point of application of the resultant of these two forces. Determine the magnitude and direction of this resultant force by the

parallelogram of forces. In a similar manner combine this resultant with one of the other forces, and continue the process until the resultant of all the forces has thus been determined in direction, point of application and magnitude. This process may be continued indefinitely if the forces are not parallel, but fails for parallel forces, since for such forces it gives only the magnitude and direction of the resultant, the point of application being indeterminate. This method is simple in its application, but the fact that it is not applicable to the case of parallel forces and that it does not give compact diagrams makes the following general method more desirable:

Let the force F_1 be resolved into any two components, such as *OP* and *P1* of Fig. 261, and let the force F_2 be resolved into the two components *O'P'* and *P'1'*. Since the effect of any force is equal to that of its components it is evident that *OP* and *P1* may be substituted for F_1 and *O'P'* and *P'1'* for F_2 without changing the result, hence the resultant of F_1 and F_2 equals the resultant of the four components *OP*, *P1*, *P'1'*, and *O'P'*.

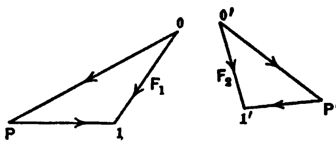


FIG. 261.

Since F_1 and F_2 may be resolved into components at any point and in any direction, and since *P1* and *P'1'* may be made parallel it is evident that *P1* and *P'1'* may be made to coincide in direction. If they can also be made equal, then the resultant of F_1 and F_2 equals the resultant of *OP* and *O'P'* and acts in the same direction. The components corresponding to *P1* and *P'1'* will be equal, parallel, and opposite in direction if the forces F_1 and F_2 be resolved, as shown in Fig. 262, in which F_1 and F_2 are given in direction and magnitude but not in position, *P* being taken at any convenient point.

In Fig. 263 the forces are shown in their correct positions and the components *OP*, *P1*, *1P*, and *P2* are plotted so that *P1* and *1P* coincide in position and are equal in magnitude and opposite in direction, therefore the resultant of F_1 and F_2 acts at the intersection of *OP* and *P2*, its magnitude and direction being given by the side *O2* of the triangle of forces, *O12*, in Fig. 262.

Had the forces F_1 and F_2 been parallel the same method could have been used, as is illustrated by Fig. 264.

In this method the point P is called the pole, *012* the force polygon, the figure *abcd* the funicular or equilibrium polygon, the lines *P0*, *P1* and *P2* connecting the pole and the apices of the force polygon the rays, and the corresponding lines in the funicular polygon the strings. The force polygon serves to determine the *magnitude* and *direction* of the resultant while the funicular polygon fixes its *position* by determining a point on its line of

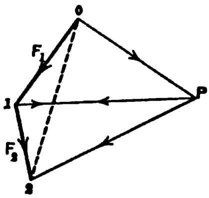


FIG. 262.

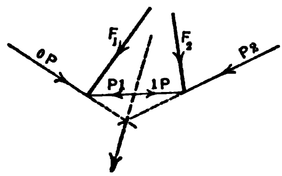


FIG. 263.

application. It is evident that the method is simple, compact, and perfectly general. The following brief statement of the method may now be made.

To find the resultant of a series of co-planar forces, lay off the forces F_1, F_2, . . . , F_n to any desirable scale, thus forming

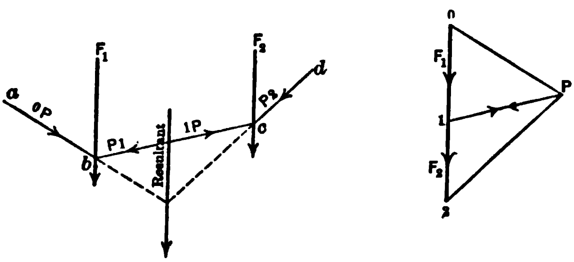


FIG. 264.

the force polygon, locate the pole P at any desirable point, draw the rays *P0*, *P1*, . . . , *Pn*, and the strings *P0*, *P1*, . . . , *Pn* parallel to these rays. In constructing these strings draw *P0* till it meets F_1, *P1* till it meets F_2, *P2* till it meets F_3, etc., each string being drawn from the point of intersection of the previous string and the appropriate force. The resultant will act through the point of intersection of the first string *P0* and the last string *Pn* and will be given in magnitude and direction by *On* of the force polygon. If the forces are in equilibrium,

the force polygon must be a closed figure, i.e., point 0 and point n must coincide, since only under this condition can $\Sigma H = 0$ and $\Sigma V = 0$. The funicular polygon must also close, that is, the string $P0$, and the string Pn must coincide, since otherwise the resultant force which equals the resultant of the two components represented by these strings would be a couple. For concurrent forces, i.e., forces all of which meet at a point, closure of the force polygon is sufficient to show that equilibrium exists, since such forces are in equilibrium if $\Sigma H = 0$ and $\Sigma V = 0$, a condition which obviously exists if the force polygon closes.

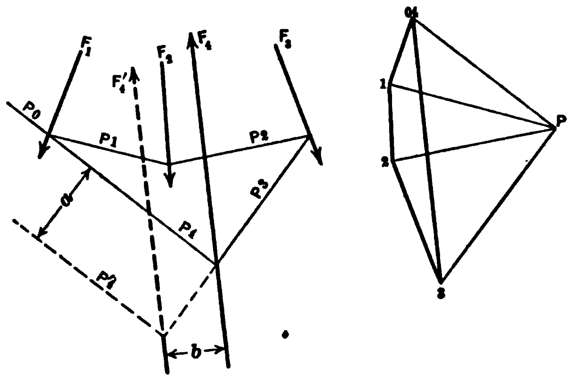


FIG. 265.

It is clear that unless the pole be located on the line On the coincidence of the first and last strings of the funicular polygon will involve the closure of the force polygon. This is illustrated by Fig. 265, in which the funicular polygon, shown by full lines, closes, since $P0$ and $P4$ coincide, a result which obviously would not occur if 0 and 4 in the force polygon were not to coincide. Were the forces in this case to consist of F_1,F_2, and F_3 only, the first and last strings of the funicular polygon could not coincide unless the pole were to be located on the line 03, or in the more general case, on the line On. If F_4 were to act in the direction indicated by the dotted line marked F'_4, the force polygon would close as before, but the last string of the funicular polygon would not coincide with the first string, but would instead have the dotted position $P'4$, and the resultant of the forces $0P$ and $P'4$ would be a couple with a value of $(P0)a = (F_4)b$.

166. Characteristics of the Funicular Polygon. The strings of the funicular polygon represent the bars of a framework which would be held in equilibrium by the applied forces, and in all of which the stresses would be either direct tension or direct compression. An infinite number of such frameworks can be selected, their position and shape being determined by the location of the pole.

Since each intermediate string represents two forces which are equal in magnitude and opposite in direction, the resultant of all the forces will be held in equilibrium by the forces represented by the extreme strings, hence this resultant acts at the point of intersection of the extreme strings, as has already been stated. The resultant of any set of consecutive forces is also held in equilibrium by the extreme strings corresponding to that particular set of forces, hence it acts at the point of intersection of these extreme strings. This may be illustrated by Fig. 265, in which the resultant of F_1 and F_2 acts at the intersection of $P0$ and $P2$; of F_1, F_2, and F_3 at the intersection of $P0$ and $P3$; of F_1, F_2, F_3, and F_4, at the intersection of $P0$ and $P4$, that is, at any point along $P0$ or $P4$, an obviously correct conclusion, since the resultant of F_1, F_2, F_3, and F_4 equals zero, these forces being in equilibrium.

The following general rule may be applied to the determination of the point of application of the resultant:—The resultant of any set of consecutively numbered forces acts through the point of intersection of the two strings, one of which is designated by a number equal to that of the highest numbered force, and the other by a number one lower than the lowest numbered force. For example, the resultant of a series of forces, F_4 to F_7 inclusive, acts through the point of intersection of $P3$ and $P7$. In applying this rule the forces and strings should be numbered in the exact manner used in the illustrations.

167. Reactions. Since in order that a set of forces may be in equilibrium the force and funicular polygons must close, it is evident that the reactions of a given structure may be determined graphically if their values are such to make these two polygons close. The method of doing this is clearly shown by the following examples.

Problem. Determine by use of the funicular polygon the reactions for the beam shown by Fig. 266.

Solution. In order that the funicular polygon may close, the first and last strings must lie on the same line. To insure that such will be the case draw the string $P0$ through the point of application of the left reaction, since this is the only known point on the line of action of this reaction.

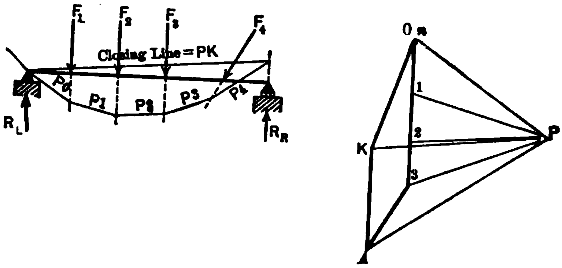


FIG. 266.

Draw the remainder of the funicular polygon in the usual manner, and draw also the line marked "closing line," which should connect the point of intersection of the string $P4$ and the reaction R_R, with the point of intersection of $P0$ and R_L. The first and last strings $P0$ and Pn of the funicular polygon may now be made to coincide by drawing the line PK in the force polygon parallel to the closing line,

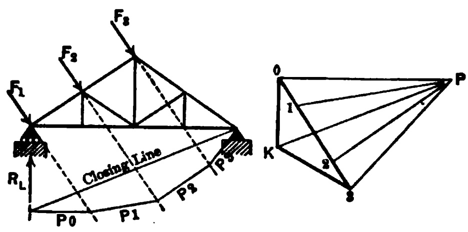


FIG. 267.

and fixing the position of K by drawing from 4 a line parallel to the reaction the direction of which is known, that is, to R_R. $4K$ will equal the right-hand reaction and $K0$ the left-hand one, since by using these as applied forces and drawing the funicular polygon for the six forces F_1, F_2, F_3, F_4, R_R, and R_L, the first and last strings will coincide.

Problem. Determine by the funicular polygon the reactions for the truss shown by Fig. 267.

Solution. For this case the left reaction is the one that is known in direction. The funicular polygon has therefore been constructed by drawing $P3$ first, thus reversing the usual method of construction. The closing line is drawn from the right point of support to the intersection of $P0$ with the left reaction. The value of the right reaction is given by $3K$ and of the left reaction by $K0$.

Further examples need not be given, as no new methods are required. The essential thing for the student to grasp is that the closing line should connect the points of intersection of the reactions and the

extreme strings (that is, the strings holding the resultant of the applied loads in equilibrium), and that the point K in the force polygon should be so located as to enable the reactions and forces in the force polygon to be read consecutively beginning with the left-hand force (or reaction). Each reaction may be identified by observing that the rays in the force polygon between which it lies correspond to the strings of the funicular polygon intersecting on its line of action.

168. Graphical Method of Moments. It is evident that the moment of any set of forces about a given axis may be obtained by scaling the distance from the axis to the line of action of the resultant of the given forces and multiplying this value by the resultant. To illustrate: Let point a (Fig. 268) represent the trace of the axis and let the problem be to find the moment about a of the forces F_1, F_2, and F_3. The resultant of these

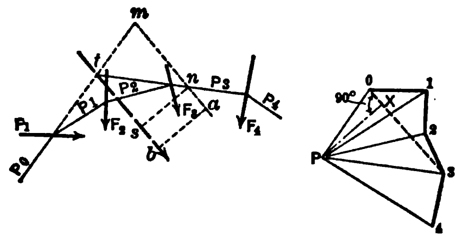


FIG. 268.

forces $= 03$, and it acts through t, the point of intersection of $P3$, and $P0$ produced, hence its moment about $a = 03$ to the scale of force multiplied by ab to the scale of distance.

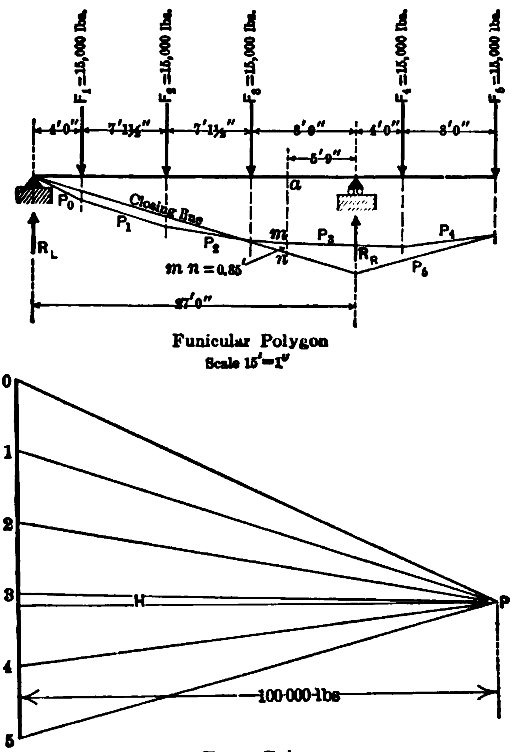
The above method is very simple, but the following modification of it is more useful. Draw through a a line, ma, parallel to 03. Then the moment about a of the given forces equals 03 multiplied by ns, the distance from ma to the line of action of the resultant. Draw from P in the force polygon a line PX perpendicular to 03. Then the $\triangle\ ntm$ is similar to $\triangle\ P\,30$;

$\therefore\ \dfrac{PX}{ns} = \dfrac{0\,3}{mn}$ (altitudes of two $\triangle$s are to each as their bases).

$\therefore\ (PX)(mn) = (03)(ns) =$ moment desired. The theorem thus deduced may be stated as follows:

To find the moment about any point of any number of coplanar forces, construct a funicular polygon corresponding to a force polygon having forces laid off consecutively, draw through the point a line parallel to the resultant of the forces, and find its intercept between the strings holding the resultant in equilibrium.

This intercept measured to the scale of distance multiplied by the perpendicular distance, hereafter called H, from the pole of the force polygon to the resultant of the given forces measured to the scale of force equals the desired moment. For a horizontal beam carrying vertical loads this equals the product of the intercept of the vertical ordinate through a and the horizontal pole distance



Force Polygon
Scale 40.000#=1"

FIG. 269.

It follows that for such a beam the funicular polygon corresponds to a curve of moments for the given loads, referred to the closing line as an axis, the moments being measured by vertical ordinates.

The character of the moment can usually be determined by inspection. If doubt exists the point of application of the resultant of the forces on one side of the section should be located, and with this known and the direction of the resultant given in

the force polygon, the character of the moment can be easily seen. For a horizontal beam with vertical loads, the fact that the moment is zero wherever the closing line intersects the funicular polygon and hence changes sign at such a point, may often be used to advantage in determining the character of the moment, as is illustrated in the problem which follows.

Problem. Determine by the graphical method of moments the bending moment at section a of the beam shown in Fig. 269.

Solution. This problem involves the determination of the moment of the forces R_L, F_1, F_2, and F_3 about a horizontal axis passing through a. Since the forces are all vertical, draw through a a vertical line and measure to the scale of distance its intercept between the strings holding the given forces in equilibrium. These are the closing line and $P3$, hence the moment equals the product of mn to scale of distance and H to scale of force. The result thus obtained equals $0.85 \times 100{,}000 = -85{,}000$ ft.-lbs. This is negative, since it is of the same character as the moment in the overhanging end, the point of zero moment occurring to the left of point a.

169. Graphical Method of Moments with a Concentrated Load System. The application of the graphical method of moments for a system of moving wheel loads may be easily made as follows:

Lay off the forces to any convenient scale and locate the pole of the force polygon so that its normal distance from the force line measured to scale of force equals some even number, say 100,000 lbs. Plot the loads to any convenient scale, and draw the funicular polygon considering the uniform load as equivalent to a series of equal concentrated loads equally spaced. If the given load system is likely to be used for a number of spans the funicular polygon should be made comprehensive enough to permit its use for any span likely to be investigated. Such a funicular is shown by Fig. 270, the force polygon being omitted.

With the funicular polygon constructed, the operation of finding the moment with any load at a given point of an end-supported span is very simple. Suppose it be desired to determine the moment at the centre of a 60 ft. span with load F_{13} at the centre.

Lay off on the funicular polygon the distance 30 ft. to the left of F_{13}, and an equal distance to the right of the same load, and draw verticals to intersect the funicular polygon at s and t.

The ordinate, mn, to the scale of distance multiplied by the distance, H, in the force polygon equals the desired moment. The moment thus obtained $= 19.2 \times 100{,}000 = 1{,}920{,}000$ ft.-lbs. In this manner several loads may be tried, and that giving the largest ordinate will give the maximum moment at the centre of this span. This method may be very conveniently used to verify the results of analytical computations, and a diagram once prepared for a standard loading, like Cooper's E_{60}, and a long span, should be of material value to the designing engineer.

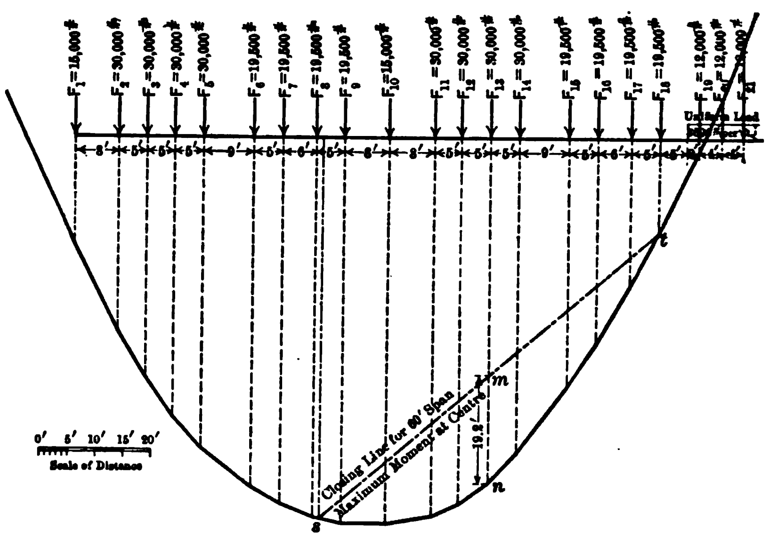


FIG. 270.—Funicular Polygon for Cooper's E_{60} Loading. Force Polygon Omitted. H of force Polygon = 100,000 lbs. Loads are Wheel Loads.

170. Graphical Method of Shear. Since the shear at any section of a beam equals the resultant, parallel to the given section, of the forces on either side of the section, its value may be determined graphically by the force polygon, the reaction having previously been determined by the method of Art. 167. The following method, however, is somewhat better adapted to the treatment of concentrated load systems and should be thoroughly understood.

Consider the beam shown in Fig. 271, and let the problem be to determine the shear at a distance x from the left end with

the first load of a concentrated load system at x. Draw the force and funicular polygons in the usual manner making PO horizontal for convenience, prolong the string PO, and draw the vertical, bc. Then the $\triangle\ abc$ of the funicular polygon is similar to the $\triangle\ POK$ of the force polygon; hence

$$\frac{bc}{ab}=\frac{KO}{PO}. \qquad \therefore\ bc=\frac{(ab)(KO)}{(PO)}$$

$$\text{but} \quad \frac{ab}{PO}=\frac{L}{H},$$

hence if H be made equal to L, bc will equal KO. But $KO=R_L$ equals the shear at x with F_1 at x, hence the following theorem may be stated:

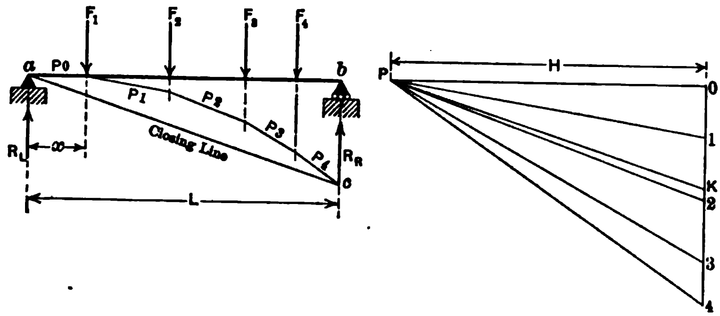


FIG. 271.

For a simple beam supported at the ends and loaded with vertical loads the shear at a distance x from the left end with the first load at x equals the vertical ordinate measured to the scale of *force* between the funicular polygon and the first string produced at a distance $L-x$ from the first load *provided the pole be so chosen as to make H and L equal.* This latter condition may be readily secured by constructing the force polygon with the loads at one point of support and the pole at the other. The vertical ordinate between the first string, PO, and the funicular polygon at a distance, $L-x$, from the first load has the same value whether the loads, force, and funicular polygons are laid off, as in Fig. 271, or as in Fig. 272, since one of these equilibrium polygons may be superposed upon the other, if drawn to same scale, by inverting it

and turning it end for end. If, therefore, the given loads are laid off, as in Fig. 272, the shear at a distance x from the left end of a simple end-supported beam may be found when the *first load is at* x by laying off the distance $L-x$ to the *left* of F_1 and scaling to the scale of force the ordinate between PO and the funicular polygon. This is equivalent to scaling the ordinate between PO and the funicular polygon at the section of the beam where it is desired to determine the shear. In order to determine the maximum shear at a given section due to a concentrated load system it may be necessary to try several loads at the section. If load (2) be moved up to the section the left reaction should be determined in the manner previously stated: i.e., by measuring the ordinate at the first load which will now be at the distance $x-d$ from the left end of the beam. The shear may now be obtained by subtracting the first load from this reaction. The shear may be found directly by scaling the ordinate between the funicular polygon and a horizontal drawn through (1) of the force polygon. This method is applicable only if the first load remains on the span. If F_1 goes off the span during the process of moving up, the ordinate should be measured at the distance $L-x$ from F_2, but the intercept should be the vertical distance between string $P1$ and the funicular polygon, since the values of vertical ordinates are in nowise affected by the fact that the first string is inclined rather than horizontal, this corresponding merely to a change in the vertical position of the pole and not a change in its horizontal position. If floor beams are used the shear must, of course, be determined by subtracting from the reaction the proper percentage of the loads between the support and the panel point at which the load under consideration is located. If the load system is to be used for a number of spans the diagram should be drawn for the longest span, and the scale for any given span determined by proportion. The application of the graphical method of shear is clearly shown by Fig. 273.

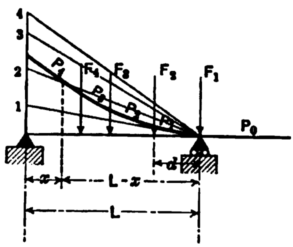
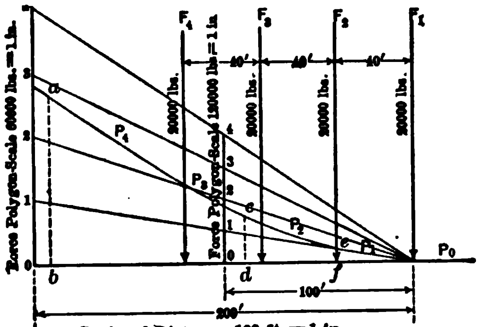


FIG. 272.

It should be noticed that the funicular polygons will be exactly alike, whatever the span chosen, *provided* the ratio of the scales of forces be inversely proportional to the spans, e.g., the funicular

polygon of Fig. 273 is constructed for a 200 ft. span to a scale of forces of 60,000 lbs. = 1 in., but if constructed for a 100 ft. span with a scale of forces of 120,000 lbs. = 1 in. as also indicated, the same equilibrium polygon would be obtained. It follows that this polygon may be used for any span by multiplying the scaled ordinate by the ratio between the span for which the polygon is constructed and the given span.



Scale of Distance 100 ft. = 1 in.

Shear, F_1 at 10 ft. from left end of 200′ span = ab = [illegible] lbs.
" F_1 " 10 " " " " " 100′ " = cd × 2 = 30000 lbs.
" F_1 " 10 " " " " " 50′ " = ef × 4 = 16000 lbs.

FIG. 273.

171. Funicular Polygon through Several Points. It is clear that if a hinged framework be constructed identical in shape to a given funicular polygon and with its bars designed to carry the string stresses of the polygon, it will be in equilibrium under the loads for which the funicular polygon is drawn. This fact may be used in determining the proper outline for a voussoir arch intended to carry a set of fixed loads, but cannot be used for moving loads. On the other hand, if a funicular polygon cannot be drawn within the limits of an arch the shape of which has already been decided upon, it is reasonable to suppose that such an arch will not be stable, and it is upon this hypothesis that the commonly accepted theory of stone arches is based. The theory of such arches will not be taken up at this point, but since in studying them it is often useful to be able to draw a funicular polygon through certain fixed points, methods of doing this will be derived.

Funicular Polygon through One Point. Since the pole may be chosen anywhere, and any string of the funicular polygon drawn

through a given point, it is evident that an infinite number of funicular polygons may be drawn through one point.

Funicular Polygon through Two Points. Since the first and last strings of a funicular polygon must always meet on the line of action of the resultant of the set of forces for which the polygon is drawn, it is evident that a funicular polygon may be drawn through two points by first drawing any funicular polygon through the first point and plotting the resultant of the set of forces in direction and position; the appropriate string of the desired polygon may then be constructed through the second point in question and the intersection of the resultant and the first string. This method is illustrated by Fig. 274, in which the

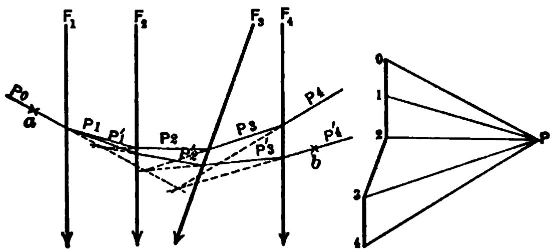


FIG. 274.

original funicular polygon is constructed with a pole located at random, the string *P0* being drawn through point *a*. The final funicular polygon may then be constructed so that *P'4* will go through point *b*, *P0* remaining unchanged in position, by drawing *P'4* through point *b* and the intersection of *P0* and *P4*. *P'3* may next be constructed by connecting the point of intersection of *P'4* and F_4 with the point of intersection of *P0* and *P3*, and in a similar manner *P'2* and *P'1* may be located. This construction is evidently consistent with corresponding strings of the two funicular polygons intersecting on the resultant of the forces held in equilibrium by these strings. If the intersections as obtained by the method just given are not on the sheet the second funicular polygon may be constructed by locating a new pole *P'*, and constructing an entirely distinct polygon. The method of doing this is illustrated by Fig. 275, in which the line of action of the resultant of all the forces is plotted and the new strings *P0* and *P'4* drawn at random through points *a* and *b* respectively to meet at

any point upon the line of action of this resultant. The new pole may then be located by drawing from *0* and *4* in the force polygon rays parallel to *P′0* and *P′4* respectively, their intersection locating the new pole *P′*. If it be desired to have other than the first and last strings pass through the two points, the resultant of the forces held in equilibrium by the desired strings should be used.

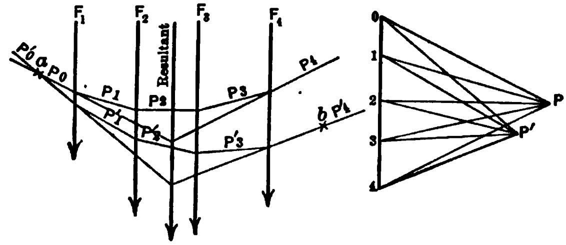


FIG. 275.

The following important theorem is also of use at times, viz.: For any set of loads, the intersection of corresponding strings of two funicular polygons drawn with different poles will lie on a line parallel to the line joining the poles. To prove this consider the two equilibrium polygons, shown in Fig. 276, with the poles *P* and *P′*.

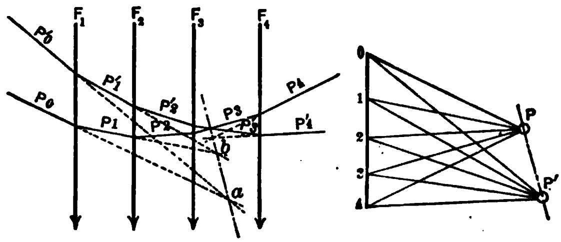


FIG. 276.

The force F_1 may be *resolved* into the two components *OP* and *P1*, consequently that force may be replaced by these components without changing existing conditions. The force F_1 will be *held in equilibrium* by the two forces *1P′* and *P′0*, consequently the resultant of these two forces is equal and opposite to the resultant of the two forces *OP* and *P1*, hence the forces *OP*, *P1*, *1P′*, and *P′0* are in equilibrium, therefore the resultant of *OP* and

$P'0$ is equal and opposite to the resultant of $P1$ and $1P'$. The resultant of $0P$ and $P'0 = P'P$ and acts at the intersection, a, of the strings $P0$ and $P'0$; the resultant of $P1$ and $1P' = PP'$ and acts at the intersection b, of the strings $P1$ and $1P'$. Since these resultants are in equilibrium they must not only be equal but act along the same line, that is, both must act along the line ab, hence ab must be parallel to the actual direction of these forces; that is, to PP'. In the same way it may be shown that the resultant of $P1$ and $1P'$ acts in the same line but in the opposite direction to the resultant of $P'2$ and $2P$, this direction being parallel to PP', hence the point of intersection of $P'2$ and $2P$ also lies on the prolongation of the line ab. In the same manner the intersection of all the corresponding strings may be proven to be on the line, ab, hence the theorem is proven.

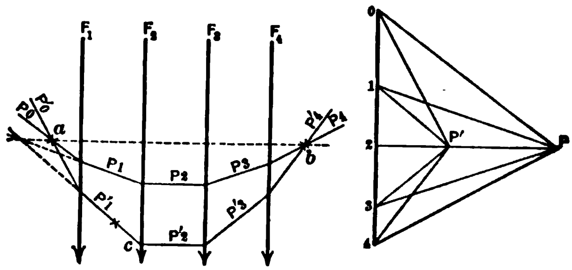


FIG. 277.

Funicular Polygon through Three Points. Application of the theorem just stated enables a funicular polygon to be passed through three points. The following is the mode of procedure: Construct a polygon through either two of the given points, say, the points a and b, Fig. 277, and connect these points by a straight line. If this line be made the line of intersection of the corresponding strings it is evident that if a new polygon be drawn it will also pass through the points a and b, hence it is merely necessary to draw a new string through the third point, c, and the intersection of the corresponding string of the first polygon with the line ab, and finish the polygon by the method given for a polygon through two points. The figure shows the application of this method, assuming that the polygon $P0$, $P1$, etc., has already been drawn through two points.

It will be noted that in the construction it was necessary to locate the pole P' in order that $P'2$ might be drawn, since the

intersection of $P2$ and the line ab does not come on the sheet. The remainder of the polygon was drawn by using the intersections of the strings of the original polygon and the line ab.

Alternative Method for Funicular Polygon through Two and Three Points. Another simple and useful method of drawing a funicular polygon through two points, a and b, is as follows: Assume the forces which are held in equilibrium by the strings which are to pass through the given points to act upon a beam supported at a and b, Fig. 278. Assume the reaction at b to be fixed in direction, vertical in this case, and determine graphically

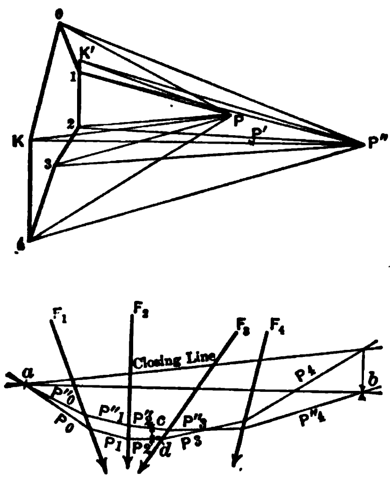


FIG. 278.

both reactions for these loads, drawing one of the strings through a. The funicular polygon thus drawn will pass through point a by construction and the value of the reactions will be *independent* of the *position* of the funicular polygon. If a new funicular polygon be now drawn with its pole at *any* point on a line drawn from the closing point K of the force polygon parallel to the line ab and with the same string passing through a, as in the original polygon, this polygon will pass through both points, since its closing line passes through a, and is parallel to ab. This construction is illustrated by the figure in which the pole P' of a funicular polygon with the string $P'0$ passing through a and $P'4$ through b might lie at any point on a line drawn from K parallel to ab.

To extend this method to a third point, c, proceed as follows: Draw a vertical through c till it intersects the original funicular polygon at d. From the pole P draw a line parallel to ad to its intersection, K', with a vertical through 2. Draw from K' a line parallel to ac. The intersection of this line with the line KP' locates P'' which is the pole of a funicular polygon passing through a, b, and c.

This method consists essentially of the determination of the reactions upon a beam, ac, due to the forces F_1 and F_2, and the location of the closing point K' of the force polygon corresponding to these reactions. A funicular polygon with its pole at any point on a line from K' parallel to ac will pass through points a and c. Since, as has already been shown, a funicular polygon with its pole at any point on KP' will pass through a and b, it is evident that a polygon with its pole at the intersection of these two lines will pass through the three points a, b, and c.

Problems

61. Determine graphically by the funicular polygon method the bending moment at a section located 6 ft. to left of left support for each beam shown in Problems 6 to 10, inclusive, page 71.

62. Construct a funicular polygon passing through the three hinges of the arch shown in Prob. 60, page 284.

CHAPTER XIV

DEFLECTION AND CAMBER

172. Elastic and Non-elastic Deflection. The deflection of a truss is due to changes in length of the members and may be divided into two parts, elastic and non-elastic. The former may be caused by stresses, or by differences in temperature of the various members, and disappears upon the removal of the loads or the return to uniform temperature; the latter is due to play at the joints and occurs when the falsework used in erection is removed, being particularly important for pin connected trusses in which considerable play usually exists in the pin holes.

A knowledge of the deflection is often desirable, particularly in proportioning the lifting devices at the ends of a swing bridge and in planning the erection of cantilever structures. A method based upon deflections also furnishes a convenient mode of determining stresses in a statically indeterminate structure.

173. Truss Deflection. Trigonometrical Method. A rigid truss is generally composed of triangles all the properties of which may be determined if the lengths of the three sides are known. The vertices of the triangles coincide with the joints of the truss, hence the various positions of a joint with respect to a pair of rectangular axes may be determined for any length of the sides of the triangles, that is, of the members of the truss. It is evident, therefore, that to find the vertical elastic deflection of a joint, say the centre joint of the bottom chord, it is only necessary to compute by trigonometry its position with respect to a fixed horizontal axis both before and after the application of the load causing the deflection. If the axis passes through the original position of the joint, the vertical movement under the load will be found without the former computation. The horizontal movement of any joint may be determined in a similar manner, using a vertical instead of a horizontal axis for reference.

The length of each member after the application of the load may be found by adding to its original length the change of length due to tension, or by subtracting the change of length due to compression.

If the non-elastic deflection be desired the same method may be used, but for this case the change in length of a member will equal the average play in the pin holes at the two ends instead of the change due to stress.

While this method is simple in theory and application, it is very laborious and is not used in practice.

174. Truss Deflection. Method of Rotation. The deflection of any joint of a simple truss due to a change in length of one bar only may be readily determined by investigating the resulting rotation of one portion of the truss with respect to the other,

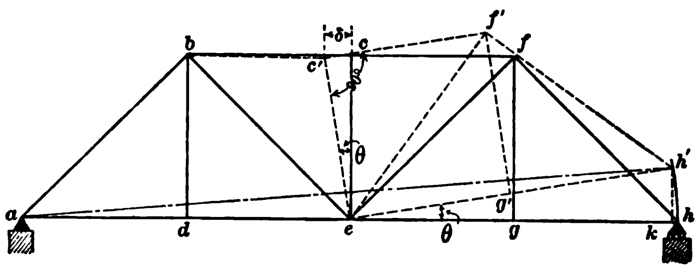


FIG. 279.

the latter being assumed as fixed. By considering the effect of each bar separately and summing up the results, the final deflection may be determined. To illustrate this, consider the truss shown by full lines in Fig. 279, and let it be desired to determine the vertical deflection, δ_e, of panel point e, due to a decrease in length, δ, of bar bc. Evidently bce is the only triangle which will be changed in shape by the change in length of this bar. If the portion, abe, of the truss be assumed for the time being to remain fixed in position, the figure $abc'f'h'ea$ will represent the new position of the truss. The value of δ for any ordinary change of length is very small compared with the original length of the bar, e.g., for a steel bar subjected to a stress of 15,000 lbs. per square inch, δ is approximately $\frac{1}{2000}$ of the length of the bar. It follows that the angular rotation of bar ec is extremely small,

and in consequence the distance cc'' in Fig. 280 may be assumed as equal to δ as shown, the error being infinitesimal.

The angular movement, θ, of bar ce evidently equals the angular movement of eh. The sine of this angle equals $\frac{\delta}{c'e}=\frac{\delta}{ce}$, hence $h'k=eh'\frac{\delta}{ce}$. But $eh'=eh$, hence $h'k=\frac{eh}{ce}\delta$ = the vertical deflection of point h with respect to the axis ae, which is assumed to remain unchanged in position. Actually, however, point h remains on the abutment and ae changes its position. The correct position of the distorted truss may be found by rotating it about a horizontal axis passing through a until ah' becomes horizontal. This rotation will cause e to drop below its original position by the amount which it is now below ah', that is, by one-half of $h'k$ (neglecting the effect of the slight difference in length between ek and eh'), hence the vertical deflection, δ_e, of point e due to the change, δ, in bar bc, will be given by the following equation:

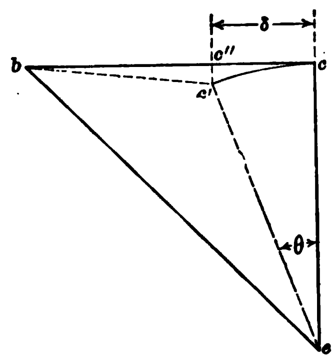


FIG. 280.

$$\delta_e=\tfrac{1}{2}\delta\frac{eh}{ce}.$$

In a similar manner the deflection of other points due to the change in length of this or other bars may be obtained.

In order to illustrate this method more completely its application to the problem of determining the vertical deflection of point e resulting from an increase, δ, in the length of bar ab, will also be given. For this case the portion, $dbfhed$, of the truss will be considered stationary. The condition of the truss after the change in length of the bar will be as shown, greatly distorted, in Fig. 281, and $e'e=\tfrac{1}{2}a'm$ will be the actual vertical upward deflection of point e.

The value of $a'm$ may be determined as follows: Let the distortion of the triangle, abd, be as shown, greatly exaggerated,

in Fig. 282. Then bn equals the new length of bar ab, and the new position of a will be at the intersection of arcs swung from b and d as centres, with radii bn and da respectively. The fact that δ is very small compared with the sides of the triangle makes

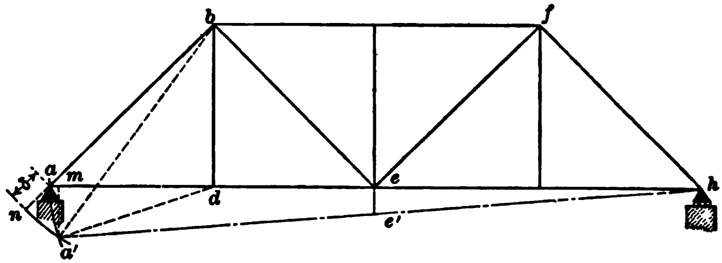


FIG. 281.

the error in assuming the tangents na' and aa' to coincide with the corresponding arcs negligible, and gives the condition shown in Fig. 282, from which it at once follows that aa' equals $\dfrac{\delta}{\sin \alpha}$. But $\sin \alpha = \dfrac{bd}{ab}$, hence $aa' = \delta \dfrac{ab}{bd}$ and therefore $ee' = \frac{1}{2}\delta \dfrac{ab}{bd}$. This method is evidently much simpler in application than the trigonometrical method previously mentioned, but is nevertheless awkward for general use, since it involves an entirely separate solution for each bar. It is given here to illustrate graphically the distortion due to change in length of a single bar.

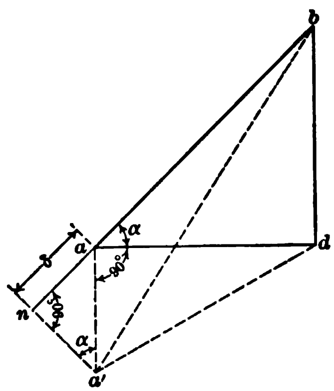


FIG. 282.

175. Truss Deflection. Method of Work. The method which follows is a modification of one usually attributed to Prof. Otto Mohr, and provides a very simple and ingenious solution of the problem of determining the deflection of a truss. The method is based upon the fact that if a truss is loaded by a *single load of unity* at any point and is then deflected by the application of *other forces*, the external work done by the load

of unity equals the internal work done by the bar stresses caused by this load of unity. This is in accordance with Bernoulli's law of virtual work, which states that a system of concurrent forces in equilibrium may be moved a small distance by an external force without the performance of work by the system. Such a condition occurs at each joint in a truss, the forces being the bar stresses due to the load of unity, and the movement of the joint being that due to the external forces producing the deflection of the truss. A slight approximation actually occurs in the application of the method, since it is assumed that the bar stresses due to the load of unity remain constant during the distortion of the structure; actually these change slightly owing to the change in the angles between the various members meeting at the joint, but the error is extremely small for trusses formed of material with a high modulus of elasticity, since the change in these angles is inappreciable for a safe working load. The method is inapplicable for a truss which would distort greatly under load, such, for example, as a truss formed of spiral steel springs or rubber bands.

It will be noticed that the load of unity is merely a measuring device and has no influence upon the deflection of the structure by the applied loads, since it and the stresses caused by it are in equilibrium before the forces causing deflection are applied, and remain in equilibrium during the application of these forces. Moreover the load of unity may be expressed in the smallest possible units, say milligrams, and could under no circumstances have an appreciable effect upon the deflections.

In order to apply this method consider the truss shown in Fig. 283 and let it be desired to determine the deflection, δ_2, of panel point L_2, due to the shortening of bar U_1U_2 by the amount δ, this shortening being due to any cause whatever, such for example as a stress in the bar, a difference of temperature in the bar as compared with other bars in the truss, or an adjustment of its length by a turnbuckle. Fig. 283 shows by full lines the truss before the length of bar U_1U_2 is changed, and by dotted lines the distorted position of the truss. Evidently the external work which would be performed by a vertical load of unity hanging at L_2 during the change of length of the bar would be unity $\times\ \delta_2$. This load would cause a compression, s, in bar U_1U_2, and this internal force would have to move the distance, δ, during the change in length of the bar, hence would perform an internal

work of $s\delta$. Equating the internal and external work gives $\delta_2 = s\delta$; therefore the vertical deflection, δ_2, of point L_2, due to a change in length, δ, of bar U_1U_2, equals the product of the stress, s, in U_1U_2, and the change in length of the bar. Were the load of unity inclined instead of vertical, s would have a different value, and δ_2 would be the deflection along the *inclined line* of action of the load of unity. A comparison of the results obtained by this method and the method of rotation shows them to be equal.

The signs must be carefully considered. If tension and increase in length are both denoted by positive signs, the deflection will be in the *direction* of action of the load of unity if the resulting value of $s\delta$ is *positive*, and in the *opposite* direction if this product is negative. For the case considered s is compression, and δ is a shortening, hence each has a negative sign and the product

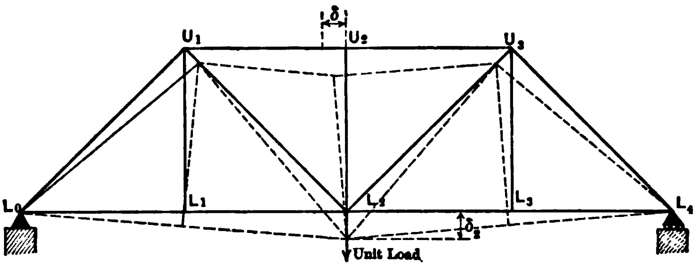


FIG. 283.

will be positive; therefore the deflection will be in the direction of action of the load of unity, that is, downward.

The correctness of this method of dealing with the signs may be readily seen by examining the case of a single vertical bar with a force of unity acting downward at its lower end. The stress, s, in this case is tension and hence has a positive sign. If the length of the bar be increased by the amount, δ, the product, $s\delta$, will also have a positive sign, showing, according to our convention, that the lower end of the bar deflects in the direction of action of the unit force; that is, downward. If the bar be shortened, $s\delta$ will have a negative sign indicating a deflection of the lower end of the bar in a direction contrary to the direction of action of the unit force; that is, upward. Both of these conclusions are obviously correct, and the *direction* of the deflection with respect to the direction of action of the unit force

would evidently be unchanged if the unit force were to be applied to the bar through a series of other bars instead of directly, and if it were to be inclined or horizontal instead of vertical.

To apply the method to all bars of a truss it is only necessary to obtain the summation of the various products. The final formula for deflection may then be written,

$$\delta_n = \Sigma s\delta,$$

in which δ_n = the component of the deflection of any joint, n, of the truss in any desired direction;

s = stress in any bar of the truss due to a load of unity acting at joint under consideration and in the direction of the desired deflection;

δ = change in length of the bar of the truss in which the stress s occurs;

$\Sigma s\delta$ = summation of the products, $s\delta$, for all bars of the truss.

If $\Sigma s\delta$ is found to have a positive value the deflection will be in the direction of action of the force of unity; if a negative value, it will be in the opposite direction. If the actual deflection of a given joint is desired the deflection in both a horizontal and vertical direction must be obtained and the resultant found.

The usual problem is to determine the deflection in a given direction of a given joint due to applied loads such as the weight of the structure itself, or a given position of the live loads. For this case δ will be the change in length due to the stress caused by the applied loads, hence the formula may be written

$$\delta_n = \Sigma s\frac{PL}{AE},$$

in which δ_n = deflection in feet of any joint in any desired direction;

L = length of any bar in feet;

A = area of same bar in square inches;

P = stress in pounds in same bar due to applied loads;

E = modulus of elasticity;

s = stress in same bar due to force of unity, acting at joint under consideration and in direction of desired deflection.

If E is constant for all bars of the truss, as is usually the case, it is simpler to express the change in length of each bar in terms of E and substitute the final numerical value of E after the summation is complete.

176. Truss Deflection Illustrated. The following example illustrates clearly the application of this method.

Problem. Let the problem be the determination of the vertical deflection of point L_2 of truss shown in Fig. 284 for a uniform live load of 2000 lbs. per foot over the entire truss.

Solution. The simplest method of solution of such a problem is to prepare a table in which separate columns are assigned for the terms s, P, L, and A; for the change in length of the bar; and for the product of the change in length of the bar and the stress, s. The table on page 363 is prepared in this manner and is self-explanatory.

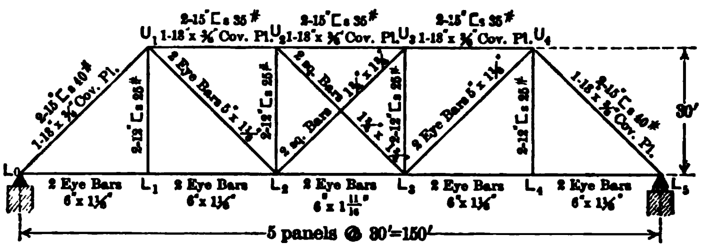


FIG. 284.

The summation of the numbers in the last column of the table gives $+\dfrac{2,170,000}{E}$, and equals the vertical deflection *downward* of L_2. Were this summation to have a negative sign it would equal the upward deflection of this point. The numerical value of the deflection may be obtained by substituting the value of E. If this be taken as 29,000,000,

$$\delta = \frac{2,170,000}{29,000,000} = 0.0749 \text{ ft.} = 090'' = \tfrac{7}{8}'' \text{ approximately.}$$

An inspection of the table shows that the stresses due to load unity should be computed before the change in length of the bars, since if the stress in any bar caused by this load is zero the deflection due to this bar is zero and its change in length need not be figured.

Had the problem been that of computing the horizontal movement of the roller-end the load of unity should have been applied horizontally at that end. The only truss bars which

TABULAR VALUES FOR DEFLECTION OF POINT L_2 OF TRUSS SHOWN IN FIG. 284

(Slide rule used throughout.)

Bar.	Stress due to Load of Unity at $L_2 = s_1$.	Stress due to Applied Load $= S_0$.	Length of Bar in Ft. $= L$	Area of Cross-section in Sq. In. $= A$.	Change in Length of Bar in Ft. $= \delta = \frac{PL}{AE}$.	Deflection due to each Bar in Ft. $= s_1\delta$.
L_0U_1	0.848(−)	169600(−)	42.4	30.27	$\frac{238000}{E}(-)$	$\frac{202000}{E}(+)$
U_1U_2	1.200(−)	180000(−)	30.0	27.33	$\frac{198000}{E}(-)$	$\frac{238000}{E}(+)$
U_2U_3	1.200(−)	180000(−)	30.0	27.33	$\frac{198000}{E}(-)$	$\frac{238000}{E}(+)$
U_3U_4	0.800(−)	180000(−)	30.0	27.33	$\frac{198000}{E}(-)$	$\frac{158000}{E}(+)$
U_4L_5	0.565(−)	169600(−)	42.4	30.27	$\frac{238000}{E}(-)$	$\frac{134000}{E}(+)$
L_0L_1	0.600(+)	120000(+)	30.0	13.50	$\frac{267000}{E}(+)$	$\frac{160000}{E}(+)$
L_1L_2	0.600 (+)	120000(+)	30.0	13.50	$\frac{267000}{E}(+)$	$\frac{160000}{E}(+)$
L_2L_3	0.800(+)	180000(+)	30.0	20.25	$\frac{267000}{E}(+)$	$\frac{214000}{E}(+)$
L_3L_4	0.400(+)	120000(+)	30.0	13.50	$\frac{267000}{E}(+)$	$\frac{107000}{E}(+)$
L_4L_5	0.400(+)	120000(+)	30.0	13.50	$\frac{267000}{E}(+)$	$\frac{107000}{E}(+)$
U_1L_1	0.000	60000(+)	30.0	un	necessary	0
U_2L_2	0.000	0	30.0		"	0
U_3L_3	0.400(−)	0	30.0		"	0
U_4L_4	0.000	60000(+)	30.0		"	0
U_1L_2	0.848(+)	84800(+)	42.4	11.25	$\frac{320000}{E}(+)$	$\frac{271000}{E}(+)$
U_2L_3	0.000	0	42.4	un	necessary	0
U_3L_2	0.565(+)	0	42.4		"	0
U_4L_3	0.565(+)	84800(+)	42.4	11.25	$\frac{320000}{E}(+)$	$\frac{181000}{E}(+)$

would be stressed by this load would be those in the bottom chord in which the stress would be 1.00 (+) were the load taken as acting to the right. The deflection would then be found by the summation of the changes in length of the bottom chord bars, which equals (+) $0.00920\times5=(+)0.046$, hence the horizontal movement of the roller-end of the truss under the load of 2000 lbs. per foot = 0.046 ft. or 0.55 in. to the right. Had it been desired to find the elastic deflection due to the dead load the dead stresses should have been used instead of the stresses due to the load of 2000 lbs. per foot.

For the non-elastic deflection due to play in the pin holes the change in length of the bar could be written directly, and the third, fourth and fifth columns omitted. For example, if the allowable play in the pin holes at L_1 and L_2 is $\frac{1}{32}$ in., the change in length of bar L_1L_2, that is, the change in distance centre to centre of pins, would be $\frac{1}{32}$ in. (+), and this value should be written in the sixth column.

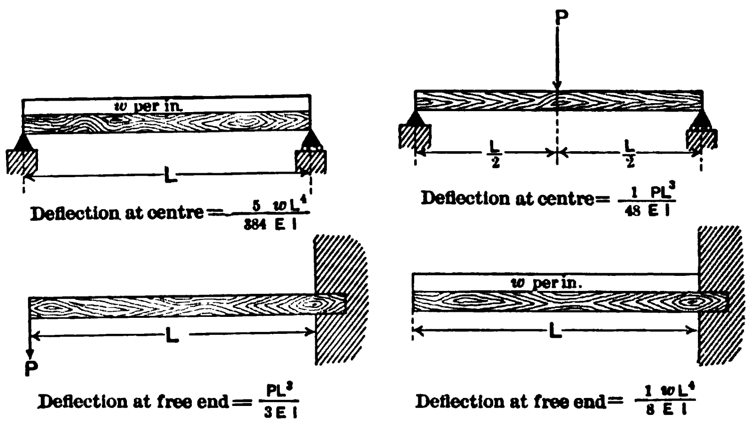


FIG. 285.

177. Deflection of Beams and Girders. Simple formulas for the deflection of beams and girders of constant cross-section supported at both ends or fixed at one end are derived in all standard works on mechanics. The more common cases are represented by Fig. 285, in which the deflections given are the maximum deflections in inches provided linear dimensions are in inches and forces in pounds. For more complicated cases of loading, or for girders with variable cross-section

the method of work may be applied in the same manner as for trusses, the fibres of the beam being substituted for the bars of the truss.

A general formula for the deflection may be developed by this method in the following manner:

Let M = moment at section, de, of the given beam (see Fig. 286) due to the external forces causing the deflection of point a;

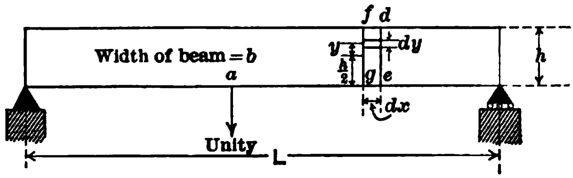


FIG. 286.

m = moment at same section due to load unity acting vertically at a;

δ = longitudinal distortion, due to the external forces, of the prism, $fdeg$ at a distance y from the neutral axis;

δ_a = deflection of point a due to the external forces;

f_1 = longitudinal fibre stress due to moment, m, at a distance y from the neutral axis;

f_2 = fibre stress at same point due to moment M;

w = internal work done in prism of length dx, depth dy, and width b, with its centre at a distance y from the neutral axis of the beam, by the load unity, during the distortion of the beam by the application of the external forces;

W = total internal work done in the beam by the stresses due to the load of unity during the beam's distortion by the external forces.

I = moment of inertia of beam in inches.[4]

Then

$$\delta = \frac{f_2 dx}{E} = \frac{My}{EI} dx,$$

and

$$w=(f_1bdy)\left(\frac{My}{EI}dx\right)$$
$$=\left(\frac{my}{I}bdy\right)\left(\frac{My}{EI}dx\right)$$
$$=\left(\frac{Mm}{EI}\right)\left(\frac{y^2bdy}{I}\right)(dx).$$

$$\therefore\ W=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}Mm\frac{y^2bdy}{EI^2}dx,$$

but

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{y^2bdy}{I}=\frac{I}{I}=1;$$

hence

$$W=\int_0^L\frac{Mm}{EI}dx.$$

The external work due to the load of unity $=1\times\delta_a$, hence

$$\delta_a=\int_0^L\frac{Mmdx}{EI}. \qquad (29)$$

The application of this equation to a beam of constant cross-section is illustrated by the following problem:

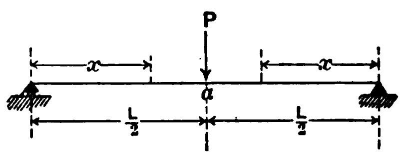


FIG. 287.

Problem. Let it be required to find the deflection at the load for the case shown in Fig. 287.

Solution. Consider a section at distance x from left end. Then

$$M=\frac{Px}{2} \quad \text{and} \quad m=\frac{x}{2}.$$

Since the beam is symmetrical the integral for the entire length of beam may be taken as double that for the left-hand half, therefore, the value of δ_a is given by the following equation:

$$\delta_a = 2\int_0^{\frac{L}{2}} \frac{Px}{2} \cdot \frac{x}{2} \cdot \frac{dx}{EI} = \frac{1}{48}\frac{PL^3}{EI}.$$

Were the beam to be loaded for its entire length with a load of p per foot, M in the above equation would be $\frac{pL}{2}x - \frac{px^2}{2}$, hence the equation for deflection would be

$$\delta_a = \frac{p}{2}\int_0^{\frac{L}{2}} \frac{(Lx - x^2)(x)dx}{EI} = \frac{5}{384}\frac{pL^4}{EI}.$$

For beams of variable section formula (29) may be applied by integrating for different portions of the beam and then adding the results. Suppose, for example, that in the first case the middle half of the beam had the value I_1 for the moment of inertia and the two end quarters each had the value I_2. The total deflection would then be expressed by the following equation:

$$\delta_a = 2\int_0^{\frac{L}{4}} \frac{Px^2}{4}\frac{dx}{EI_2} + 2\int_0^{\frac{L}{4}} \frac{P\left(\frac{L}{4}+x\right)^2}{4}\frac{dx}{EI_1}.$$

The case just given illustrates the method necessary for an end-supported girder with a single cover plate on each flange extending over the central half of the girder. If more cover plates are used it is necessary to write more terms, but the same general method is applicable. If the girder varies in depth, as well as in flange area, it may be divided into as many sections as seems desirable and the average moment of inertia of each section used, the equation for the deflection including as many terms as there are moments of inertia.

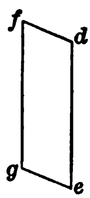


FIG. 288.

Before leaving this method it should be observed that both by it and the elastic curve method, by which the results shown in Fig. 285 are usually obtained, the deflection due to shear is neglected. The effect of positive shear at the section under consideration would be to distort the prism, $fdeg$, in the manner shown by Fig. 288, and hence cause some deflection. The value of the deflection due to shear is, however, relatively very small and may be neg-

lected. For a full discussion of this method the reader is referred to a paper by Clarence W. Hudson in the "Transactions of the American Society of Civil Engineers," Vol. LII.

178. Graphical Method of Truss Deflection. *Williot Diagram.* The method of work given in Art. 176 furnishes a simple and accurate method of determining the deflection in a given direction of a particular joint in a truss. It is, however, occasionally desirable to determine the actual distortion of several or all joints, a problem which can be solved somewhat more readily by graphical than by analytical methods.

It is obvious that the actual movement of any truss joint, due to changes in the bar lengths, may be determined graphically in the following simple manner:

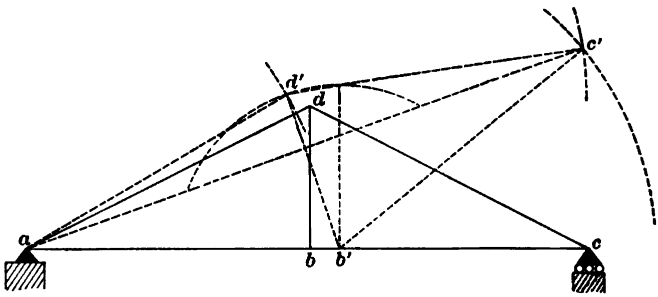


FIG. 289.

Let *abcd*, Fig. 289, be the truss before the bar lengths are changed, point *a* being fixed in position. Let the new lengths of the bars be *ab'*, *b'c'*, *c'd'*, *d'a*, and *b'd'*. Assume for the present that bar *ab* remains unchanged in direction. If its length be now changed to *ab'*, *b* will move to *b'*, and if from *a* and *b'* as centres arcs with radii *ad'* and *b'd'* be swung these arcs will intersect at *d'*, which will be the new position of *d* on the assumption that *ab* remains unchanged in direction. In a similar manner arcs swung from *b'* and *d'* with the new lengths of the other bars as radii will give the new position of *c* at *c'*, hence *ad'c'b'* will be the actual shape of the truss after distortion takes place. Its position is of course incorrect, since point *c* should remain on the abutment, hence the line *ac'* should actually be horizontal, and by so considering it the deflections of all points may be obtained; e.g., the vertical deflection of

point b equals the normal distance from b' to ac', the horizontal deflection of point $c = ac' - ac$, etc.

This method, which may be extended to cover any form of truss, is impracticable in practice, since accurate results cannot be obtained without the use of a very large scale, owing to the very small changes in bar lengths for materials having the high moduli of elasticity of structural materials. To overcome this difficulty a modification of this method by which changes of length only are dealt with was developed by the French engineer, Williot, and will now be given.

Let the truss shown in Fig. 290 be identical with that given in Fig. 289, and assume the same changes in bar lengths to occur. Assume also as before that bar ab is fixed in direction, and that b moves to b' when distortion occurs.

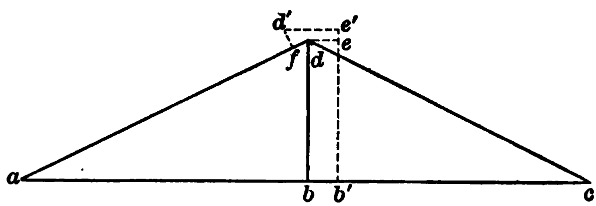


FIG. 290.

Let $b'e$ be parallel to bd and equal to it in length;
" ee' be the change in length of bd (an increase);
" df be the change in length of ad (a decrease).

If normals to ad and $b'e'$ be erected at points f and e' they will meet at d', which will be the new position of d for any rigid truss of ordinary structural materials, since the distortions are so small compared with the bar lengths that the normal $e'd'$ may be considered as coinciding with the arc swung from b' as a centre with radius $b'e'$, and normal fd' may also be considered as coinciding with the arc swung from a as a centre of with radius af. Since the figure $dee'd'f$ may be drawn to *any scale* without reference to the truss itself, as shown in Fig. 291, it is evident that the actual displacement of point d with reference to axis ab may be found with great accuracy. In a similar manner each triangle of which the truss is composed may be dealt with and the displacement of each vertex found with reference to any one of the sides of the triangle as an axis.

If this process be carried out for the entire truss each displacement may be determined with respect to bar *ab* as an axis, by using for each new axis the new position of that bar which is common to any triangle previously considered and that under consideration; e.g., to find the new position of point *c*, with reference to axis *ab*, use the new position of the bar *bd*, that is, *b'd'*, as an axis.

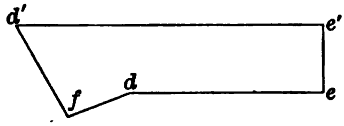


FIG. 291.

This is illustrated in Fig. 292, where *ab'd'* is the new position of the triangle, *abd*, resulting from the changes in length of bars *ab*, *bd*, and *ad*. To find the position of *c'* due to the combined distortion of the two triangles, *abd* and *bcd*, it is evidently necessary to determine the intersection of two arcs, one swung from *b'* as a centre with the new length of bar *bc*, and the other from

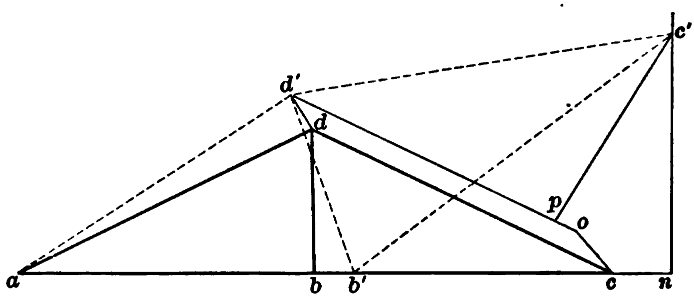


FIG. 292.

d' with the new length of bar *dc*. Here, as before, it is essentially correct to replace the arcs by their tangents. The process may be accomplished by laying off *cn* equal to the combined increase in length of *ab* and *bc*, *co* equal and parallel to *dd'*, *op* equal to the decrease in length of *cd*, and *nc'* and *pc'* perpendicular respectively to *bc* and *dc*. The point of intersection, *c'*, of these normals is the new position of point *c*.

It should be observed that the new position of point *c* is a function of the change in shape of triangle *abd* as well as of triangle *bcd*, even if bar *ab* actually remains horizontal, since bar *bd* cannot be changed in length without influencing the shape of triangle *bcd*.

Since dd' of Fig. 291 equals, numerically, co of Fig. 292, and de of Fig. 291 equals the change in length of bar ab, it is evident that Fig. 291 may be used as a basis for determining the movement of point c. The resulting diagram is called a Williot diagram and is shown in Fig. 293.

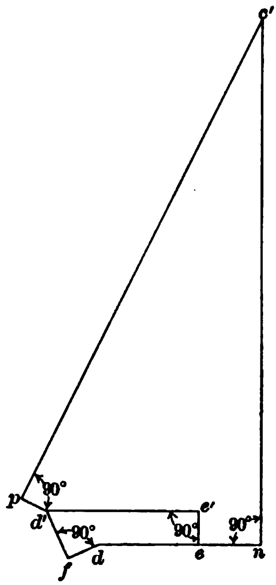


FIG. 293.

In this diagram

df = decrease in length of bar ad;
de = increase in length of bar ab;
en = increase in length of bar bc;
ee' = increase in length of bar bd;
$d'p$ = decrease in length of bar dc.

If all of above changes are plotted parallel to the original directions of the bars, then

dd' will equal the displacement of the point d,
and dc' the displacement of point c.

In order to avoid confusion it is desirable to letter the initial point the same as the panel point used for an origin, in this

case point a, hence if in the diagram letter d is replaced by a^1 the displacement will in every case be the distance from a to

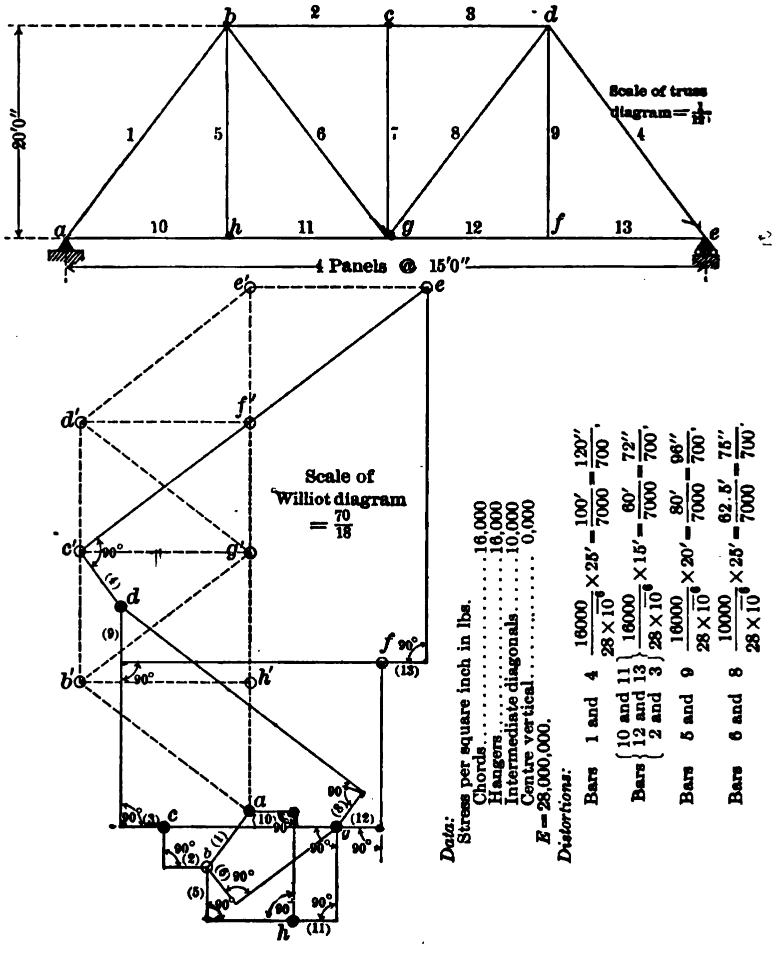


Diagram is drawn assuming a to be fixed in position and ab in direction. Distances from a to b, h, c, g, d, f, and e equal actual movements of these points referred to axis ab.
Figures in brackets indicate bar numbers and are placed on the lines which show the changes in length of the corresponding bars.

FIG. 294.

the intersections of the proper normals. A Williot diagram for a more complicated truss is shown by the full lines of Fig. 294.

The following rules for the construction of the Williot diagram may now be given:

1. Draw a diagram of the truss, and select one member as an axis, or fixed reference bar, assuming this bar as fixed in direction with one end fixed in position. Choose, if possible, a bar which does not change its direction under the given loading.

2. From some convenient point plot the axial deformation of this reference bar parallel to and in the direction of the distortion of its free end. Letter the origin and the point just plotted, hereafter called the second point, to correspond with the lettering of the fixed and free ends respectively of the chosen reference bar.

3. Select two bars which with the fixed reference bar form a triangle. From the origin, on a line parallel to that one of the two bars which is hinged to the fixed end of the reference bar, lay off the axial deformation of this second bar in the direction of the motion of its far end. From the second point, similarly lay off the deformation of the third bar. At the extremities of these plotted deformations erect perpendiculars. The distance from their point of intersection to the origin equals the distortion of the apex of the triangle under consideration *with reference to the fixed axis* and should be lettered to correspond to this apex in the truss diagram.

4. Consider the bars forming a triangle, of which the displaced positions of the ends of one leg are given by two of the three points now plotted. From each of these two points, on a line parallel to the bar of the triangle which is connected thereto, lay off the axial deformation of this bar in the direction of the motion of its far end. At the extremities of these plotted deformations erect perpendiculars. The distance from this point of intersection to the origin equals the distortion of the apex of the triangle under consideration with reference to the *fixed axis*, and should be lettered to correspond to this apex in the truss diagram.

5. Continue thus till all points are located.

179. Correction of the Williot Diagram. It is evident that the Williot diagram shows the actual movement of the joints only when the bar which is assumed to be fixed in direction actually remains fixed during the change in shape and when the origin also remains fixed. The latter condition is readily obtained by selecting a point which is not on rollers, and the former may sometimes but not always be secured, e.g., in the

truss shown by Fig. 294 if loaded with a uniform load per foot, bar cg will remain vertical while the truss deflects. As in many cases no bar remains fixed in direction this method would be incomplete unless some means can be found for correcting the displacements thus obtained to provide for the rotation about the assumed axis. If the displacements found by the Williot diagrams for truss of Fig. 294 be plotted, the truss will appear as shown by dotted lines in Fig. 295 with the distortion exaggerated owing to the plotting of the displacements on a different scale from the truss

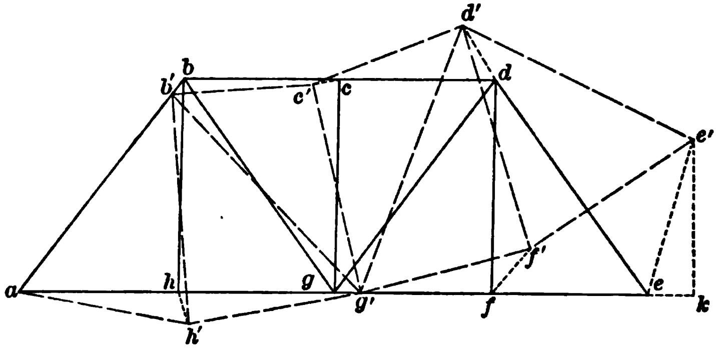


FIG. 295.

diagram. Since point e should remain on the abutment, its true movement being horizontal, the correction necessary to apply to the diagram must be such as would be produced by rotating the whole truss about a until e' drops to the horizontal line through a, that is, until e drops through the distance $e'k$ (since here again the arc swung with a as a centre and ae' as a radius differs in position from the tangent by only an infinitesimal amount, the correct distance $e'k$ being very small).

The movement of the other points of the truss due to this rotation will bear the same relation to the movement of e as the distance from a to these points bears to the distance ae.

In Fig. 296 all the full lines are perpendicular to the corresponding bars of the actual truss shown in Fig. 297. In consequence any triangle such as $a'd'e'$ is similar to the corresponding triangle ade, hence

$$\frac{a'd'}{ad}=\frac{a'e'}{ae}; \quad \therefore\ a'd'=a'e'\times\frac{ad}{ae}.$$

In a similar manner it may be shown that

$$a'c' = a'e' \times \frac{ac}{ae}, \quad a'b' = a'e' \times \frac{ab}{ae}, \text{ etc.}$$

Hence if $e'a'$ equals the movement of point e due to rotation, $d'a'$ will equal the movement of point d, $c'a'$ the movement of point c, etc. To obtain, therefore, the true displacement of the various points the displacements determined in Fig. 294 must be corrected by these amounts. A simple method of accomplishing this has been devised by Prof. Mohr and is illustrated by Fig. 294. It consists of the insertion in the Williot diagram of a figure corresponding to Fig. 296 with a' at a and e' on a horizontal through e. The correct displacement of a point will then be given in *direction* and *magnitude* by the distance from the corresponding point of the inserted truss, shown dotted, to the same point as located by the Williot diagram, e.g., the

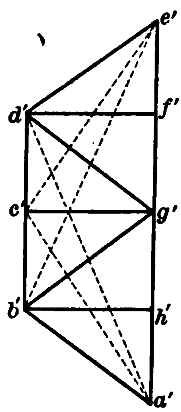


FIG. 296.

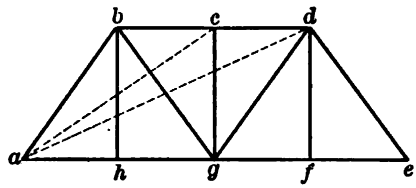


FIG. 297.

correct displacement of point $c = c'c$. The truth of this is easily seen; ac = movement shown by the diagram, $c'a$ = movement due to rotation, hence $c'c$ equals actual displacement of point c. In other words rotation causes point c to move from c' to a, and distortion from a to c, the resultant movement equalling $c'c$.

This method of correction for rotation is simple and no confusion need arise if the following rules be observed:

1. Draw a line through the displaced position, as given on the Williot diagram, of the truss joint at the expansion point of support, parallel to the known direction of movement of this point, i.e., in general, parallel to the surface upon which the expansion rollers move.

2. Draw through the point on the Williot diagram corresponding to the joint at the fixed point of support of the structure, a line perpendicular to the line in the truss connecting the two points of support of the truss, and determine its point of intersection with the line previously drawn.

3. Insert in the Williot diagram a truss diagram drawn with all its bars *perpendicular* to corresponding bars in the actual truss, i.e., drawn in a position perpendicular to the original position of the truss. The location and scale of this new truss

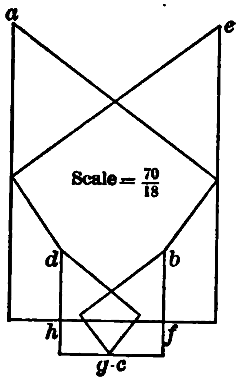


FIG. 298.

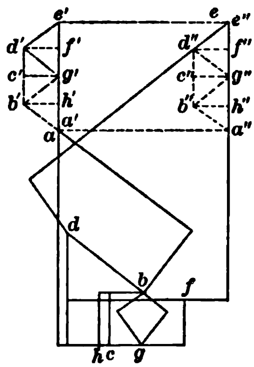


FIG. 299.

FIG. 298.—Displacement Diagrams for Truss Shown in Fig. 294. Bar *cg* assumed to be fixed in direction.

FIG. 299.—Displacement Diagram for Truss Shown in Fig. 294. Point *h* assumed fixed in position and bar *hg* in direction. Correction diagram a'', b'', d'', e'' is drawn on assumption that point *a* is free to move horizontally instead of point *e*.

diagram is fixed by locating the joint corresponding to the expansion point of support at the point of intersection previously determined (see 2) and the joint corresponding to the fixed point of support at the corresponding point on the Williot diagram.

4. The correct displacement of each joint of the truss may now be determined in magnitude and direction by scaling the distance from the joint as given on the correction diagram drawn

as described under (3) and the position of the joint as given on the Williot diagram.

Figs. 298, 299, and 300 illustrate more fully the graphical method as applied to the truss shown in Fig. 294. In Fig. 298 bar *cg* has been assumed as fixed in direction. This agrees with the actual condition if the truss is loaded uniformly and the displacement diagram needs no correction for rotation. Fig. 299 is drawn with *hg* fixed in direction and point *h* in position. This needs to be corrected for rotation and the correction diagram is given both for the truss shown, and for the same truss with point *a* free to move horizontally. Fig. 300 is drawn to show

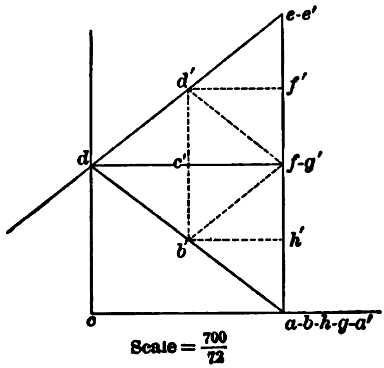


FIG. 300.—Displacement Diagram for Truss Shown in Fig. 294. Point *a* and bar *ah* assumed fixed in position and direction respectively. Change in length assumed to occur in bar 2 only. (Decrease of $\frac{80}{7000}$ ft.)

the effect of a change in length of one bar only. The correction diagram must also be drawn for this case and is shown in the figure.

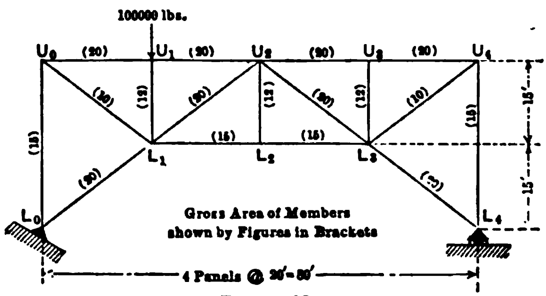
180. Camber Defined. A structure is said to be cambered when so constructed that it will not assume its theoretical form until fully loaded. For beams and short-span girders no provision for camber need be made. Girders of long span are often cambered by being slightly arched. Trusses, except in the case of comparatively short riveted spans, should always be cambered. This is particularly important in the case of pin bridges as ordinarily constructed with the splices of the top chord dependent for their strength upon the intimate contact of the planed ends of the chord sections.

181. Rules for Computing Cambers. Short span parallel chord trusses are ordinarily cambered by the following more or less empirical rule: Make the top chord panel lengths longer than those of the bottom chord by $\frac{1}{8}$ in. for every 10 ft. in length. This process necessitates a corresponding change in the diagonals, but the verticals and bottom chords are unaffected.

For long spans or for trusses with curved top chords the cambering should be accomplished by decreasing the length of the tension members and increasing that of the compression members by an amount equal to their change in length under the dead stresses and the live stresses due to full live load with due allowance for pin hole play, using for a basis the geometrical lengths of the bars, as given in the truss diagram. The lengths of the bars thus obtained correspond to the outline of the structure when assembled on false work and not carrying its own weight. When the false work is removed the structure will deflect by an amount equal to the non-elastic deflection plus the deflection due to its own weight. If cambered in this manner the application of the live load, under which the changes in length of the bars were computed, should cause the truss to take the shape of the theoretical diagram.

Problems.

63a. Compute horizontal deflection of point $L4$ of this steel structure due to the *applied load* shown, and state its direction. $E = 30{,}000{,}000$.



PROB. 63*a*.

b. Determine the magnitude and direction of the horizontal force which must be applied at L_4 to deflect point L_4 horizontally an amount equal to the deflection determined in the first portion of this problem.

64. Determine the magnitude and direction of the horizontal movement of point L_4 of above arch due to the top chord being heated 50° F. above the remainder of the structure.

CHAPTER XV

CONTINUOUS AND PARTIALLY CONTINUOUS GIRDERS AND SWING BRIDGE REACTIONS

182. Definitions. Continuous girders[1] are structures supported at more than two points, and capable of carrying bending moments and shear at all sections throughout their entire length. Such structures are commonly used for swing bridges and in reinforced concrete buildings, but their employment for ordinary bridges is inadvisable owing to the difficulty in obtaining rigid supports, a slight settlement of one pier changing materially the magnitude of all the reactions. Continuous structures are also subject to both positive and negative live moments over portions of their length and in consequence may require additional material to provide for the resulting reversal of stress; they are also stressed by changes of temperature. Partially continuous girders are structures supported at more than two points, but so built that the continuity is interrupted at one or more sections. Such girders are generally trusses in which the continuity is broken by the omission of diagonals in one panel, as was noted in connection with cantilever trusses.

183. Reactions on Continuous Girders. Method of Computation. The reactions on continuous girders can be accurately determined by the "Theorem of Three Moments," if the moment of inertia and modulus of elasticity of the material are constant throughout, a condition which sometimes exists for beams. If the moment of inertia and the modulus of elasticity of the material are not constant throughout, the reactions cannot be accurately computed until the cross-section areas are known, hence an accurate determination by this method of the stresses for such structures can only be accomplished by a series of approximations,

[1] As used in this chapter the word girder is intended to cover all cases of continuous structures and is used indiscriminately for beams, plate girders, and trusses.

the reactions first being approximately determined, the stresses and areas computed, and the computations revised to correspond to the new areas, the process being repeated as often as is necessary to obtain sufficiently precise results. A common custom, however, for continuous girders is to design the structure on the assumption that the moment of inertia is constant throughout, the "Three-moment Equation," derived from the differential equation of the elastic line being used to determine the reactions.[1]

184. Derivation of the "Three-moment Equation." Let the girder shown in Fig. 301 have n spans. There will then be

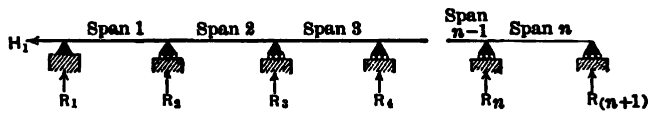


FIG. 301.

$(n+2)$ unknown reactions, all the supports but one being on rollers, and hence $(n-1)$ equations must be obtained from other conditions than those of statics. These equations may be deduced from the differential equation of the elastic curve, $\dfrac{d^2y}{dx^2}=\dfrac{M}{EI}$, by the method which follows, each of the $n-1$ resulting equations connecting the moments at three adjoining supports.

Let Fig. 302 represent a portion of a continuous girder with

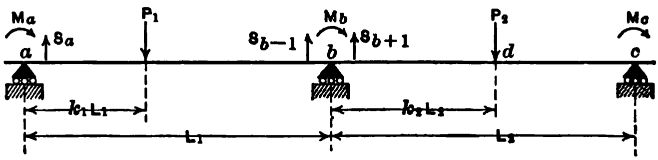


FIG. 302.

a constant moment of inertia and modulus of elasticity, the entire structure having n spans, and the section under consideration including any portion of the girder supported upon three adjoining supports. The axis of the unloaded beam is assumed to be straight and the supports level. The assumption that each load acts at a distance, kL, from the adjoining support, is adopted from Merriman's "Mechanics of Materials," and simplifies greatly the resulting equations.

[1] See also Art. 190.

Let M_a, M_b, and M_c be the moments upon the beam at the three adjoining supports, and let these be assumed as positive, when the moment of the forces on the left of the section is clockwise.

Let S_a, S_{b-1} and S_{b+1} be the shear at infinitesimal distances from the supports, a and b, and let these be assumed as positive when acting as shown.

Let M equal the moment at any section of the girder.

Let t_c, t_{b+1}, and t_{b-1} equal the tangents at c and at infinitesimal distances on either side of b of the angle between the neutral axis of the deflected girder and its original position.

Let h=normal distances of points a, b and c above some assumed axis parallel to abc and below it.

For the portion of the girder between b and c, the moment at a distance, x, from b is given by the following equations:

$$M=EI\frac{d^2y}{dx^2}=M_b+S_{b+1}x \text{ (for portion of girder between } b \text{ and } d). \quad (30)$$

$$M=EI\frac{d^2y}{dx^2}=M_b+S_{b+1}x-P_2(x-k_2L_2) \quad (31)$$

(for portion of girder between d and c).

From (30) by integration we obtain

$$EI\frac{dy}{dx}=M_bx+\frac{S_{b+1}x^2}{2}+C_1EI, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and

$$EIy=\frac{M_bx^2}{2}+\frac{S_{b+1}x^3}{6}+C_1EIx+C_2EI. \quad . \quad . \quad (33)$$

When $x=0$,

$$\frac{dy}{dx}=t_{b+1} \quad \text{and} \quad y=h; \quad \therefore \quad C_1=t_{b+1} \quad \text{and} \quad C_2=h,$$

hence

$$EIy=\frac{M_bx^2}{2}+\frac{S_{b+1}x^3}{6}+t_{b+1}EIx+EIh. \quad . \quad . \quad . \quad (34)$$

From (31) by integration we obtain

$$EI\frac{dy}{dx}=M_bx+\frac{S_{b+1}x^2}{2}-\frac{P_2x^2}{2}+P_2k_2L_2x+C_3, \quad . \quad . \quad (35)$$

and

$$EIy=\frac{M_bx^2}{2}+\frac{S_{b+1}x^3}{6}-\frac{P_2x^3}{6}+\frac{P_2k_2L_2x^2}{2}+C_3x+C_4. \quad . \quad (36)$$

When $x=k_2L_2$, $EI\frac{dy}{dx}$ in (35) = corresponding value in (32),

$$\therefore\ C_3=t_{b+1}EI-\frac{P_2}{2}(k_2L_2)^2. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

When $x=L_2$, $y=h$,

$$\therefore\ C_4=EIh-\frac{M_b}{2}L_2^2-\frac{S_{b+1}L_2^3}{6}$$

$$+\frac{P_2}{6}L_2^3-\frac{P_2}{2}k_2L_2^3-t_{b+1}EIL_2+\frac{P_2}{2}k_2^2L_2^3.$$

Hence

$$EIy=\frac{M_b}{2}(x^2-L_2^2)+\frac{S_{b+1}-P_2}{6}(x^3-L_2^3)$$

$$+\frac{P_2}{2}(k_2L_2x)(x-k_2L_2)+t_{b+1}EI(x-L_2)$$

$$+EIh+\frac{P_2}{2}k_2L_2^3(k_2-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The value of y given by (34) equals its value as given by (38) when $x=k_2L_2$; equating these values gives

$$0=-\frac{M_b}{2}L_2^2-\frac{S_{b+1}}{6}L_2^3-\frac{P_2}{6}L_2^3(k_2^3-1)-t_{b+1}EIL_2+\frac{P_2}{2}k_2L_2^3(k_2-1),$$

hence

$$t_{b+1}EI=\frac{P_2L_2^2}{6}(1-k_2^3+3k_2^2-3k_2)-\frac{M_b}{2}L_2-\frac{S_{b+1}}{6}L_2^2.$$

But

$$M_b+S_{b+1}L_2-P_2L_2(1-k_2)=M_c \quad . \quad . \quad . \quad . \quad (39)$$

$\therefore$ by substituting for S_{b+1}, its value from this latter equation, we obtain

$$t_{b+1}=\frac{L_2}{6EI}[P_2L_2(1-k_2)(k_2-2)k_2-2M_b-M_c]. \quad . \quad (40)$$

When $x=L_2$, $\frac{dy}{dx}=t_c$; $\therefore$ the value of t_c may be obtained from (35) by placing $x=L_2$, and substituting for C_3 its value as

obtained from (37) after substituting for t_{b+1} its value from (40), and for S_{b+1} its value from (39).

The result thus obtained is given by the following equation:

$$t_c = \frac{L_2}{6EI}[M_b + 2M_c + P_2L_2k_2(1-k_2^2)]. \quad . \quad . \quad . \quad (41)$$

By working in a similar manner in span L_1 an equation identical to (41) would be obtained with the indices reduced to correspond to the nomenclature of span L_1. The equation for this span may therefore be written at once, and will be as follows:

$$t_{b-1} = \frac{L_1}{6EI}[M_a + 2M_b + P_1L_1k_1(1-k_1^2)]. \quad . \quad . \quad (42)$$

The two values, t_{b+1} given by (40) and t_{b-1} given by (42), are identical, since they equal the tangents to the slope of the neutral axis at two sections an infinitesimal distance apart, hence they may be placed equal to each other, thereby enabling the following equation to be written:

$$M_aL_1 + 2M_b(L_1+L_2) + M_cL_2 = P_1L_1^2(k_1^3 - k_1) + P_2L_2^2(3k_2^2 - k_2^3 - 2k_2). \quad (43)$$

Equation (43) is the three-moment equation in its general form and is applicable to structures of constant cross-section, homogeneous material, and supported on level supports.

To obtain the corresponding equation for uniform loads substitute the following values and integrate between proper limits:

$$P_1 = w_1dx, \quad P_2 = w_2dx, \quad k_1L_1 = x_1 \quad \text{and} \quad k_2L_2 = x_2.$$

In these equations w_1 and w_2 are the loads per foot on the two spans.

For the case where the load extends over the entire structure this gives:

$$\begin{aligned} M_aL_1 + 2M_b(L_1+L_2) + M_cL_2 &= L_1^2\int_0^{L_1} w_1dx\left[\left(\frac{x_1}{L_1}\right)^3 - \frac{x_1}{L_1}\right] \\ &\quad + L_2^2\int_0^{L_2} w_2dx\left[3\left(\frac{x_2}{L_2}\right)^2 - \left(\frac{x_2}{L_2}\right)^3 - 2\left(\frac{x_2}{L_2}\right)\right] \\ &= L_1^2w_1\left(\frac{L_1^4}{4L_1^3} - \frac{L_1^2}{2L_1}\right) + L_2^2w_2\left[\frac{L_2^3}{L_2^2} - \frac{L_2^4}{4L_2^3} - \frac{L_2^2}{L_2}\right] \\ &= -\tfrac{1}{4}L_1^3w_1 - \tfrac{1}{4}L_2^3w_2. \quad . \quad . \quad . \quad (44) \end{aligned}$$

For the case of a level beam and non-level supports the constants of integration would have contained values of h which would not have cancelled, hence the final form of the equation would have included these terms. If the original axis of the girder be slightly curved before loads are applied, and if the supports fit the curve of this unloaded girder, the results found will, however, be correct. For certain interesting properties of the three-moment equation and for information concerning its historical development the reader is referred to "Mechanics of Materials" by Merriman.

185. Application of the Three-moment Equation. For purpose of illustration the following example of the application of the three-moment equation is given:

Problem. Compute by the three-moment equation the reactions for the concentrated loads shown in Fig. 303.

Solution. First apply equation (43) to spans 1 and 2.

For these two spans

$$\begin{aligned}
&M_a = \text{moment at } R_1 = 0, \text{ since this is at the end of the girder;}\\
&M_b = \text{moment at } R_2 = M_2;\\
&M_c = \text{moment at } R_3 = M_3;\\
&P_1 = 0;\\
&P_2 = 10{,}000 \text{ lbs.};\\
&L_1 = L_2 = 10 \text{ ft.};\\
&k_2 = \frac{6}{10};
\end{aligned}$$

hence

$$2M_2(20) + 10M_3 = 10{,}000 \times 100(1.08 - 0.216 - 1.2) = -336{,}000 \text{ ft.-lbs.}$$

Now apply eq. (43) to spans 2 and 3.

For these spans:
$$\begin{aligned}
&M_a = \text{moment at } R_2 = M_2;\\
&M_b = \text{moment at } R_3 = M_3;\\
&M_c = \text{moment at } R_4 = M_4;\\
&P_1 = 10{,}000 \text{ lbs.};\\
&P_2 = 5{,}000 \text{ lbs.};\\
&L_1 = L_2 = 10 \text{ ft.};\\
&k_1 = \frac{6}{10};\\
&k_2 = \frac{5}{10};
\end{aligned}$$

hence

$$\begin{aligned}
10M_2 + 40M_3 + 10M_4 &= 10{,}000 \times 100(0.216 - 0.600)\\
&\quad + 5000 \times 100(0.75 - 0.125 - 1.00)\\
&= -384{,}000 - 187{,}500 = -571{,}500 \text{ ft.-lbs.}
\end{aligned}$$

Finally apply eq. (43) to spans 3 and 4.

For these spans M_a = moment at $R_3 = M_3$;
M_b = moment at $R_4 = M_4$;
M_c = moment at $R_5 = 0$;
$P_1 = 5000$ lbs.;
$P_2 = 0$;
$L_1 = L_2 = 10$ ft.;
$k_1 = \frac{5}{10}$;

hence

$$10M_3 + 40M_4 = 5000 \times 100(0.125 - 0.500) = -187{,}500 \text{ ft.-lbs.}$$

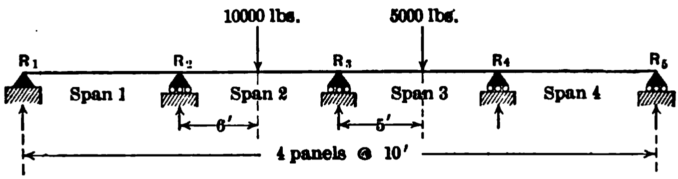


FIG. 303.

Solving the three equations thus derived for the three unknowns, M_2, M_3, and M_4, gives the following values:

$$M_2 = -\ \ 5{,}250 \text{ ft.-lbs.};$$
$$M_3 = -12{,}600 \text{ ft.-lbs.};$$
$$M_4 = -\ \ 1{,}537 \text{ ft.-lbs.}$$

But $M_2 = R_1 \times 10$,

$$\therefore\ R_1 = -525 \text{ lbs. (acting down)},$$

and

$$M_4 = R_5 \times 10,$$

$$\therefore\ R_5 = -154 \text{ lbs. (acting down)}.$$

Moreover,

$$M_3 = 20R_1 + 10R_2 - 40{,}000 = 20R_5 + 10R_4 - 25{,}000,$$

hence

$10R_2 = 40{,}000 - 12{,}600 + 10{,}500 = 37{,}900$; $\therefore\ R_2 = (+)3790$ (acting up),

and

$10R_4 = 25{,}000 - 12{,}600 + \ 3{,}080 = 15{,}480$; $\therefore\ R_4 = (+)1550$ (acting up).

Application of $\Sigma V = 0$ gives $R_3 = (+)10340$ (acting up).

In a similar manner the reactions for any number of spans, whether equal or unequal in length, and for any loading may be readily computed.

186. Reactions, Shears, and Moments for Common Cases of Continuous Girders. In order to simplify the determination of reactions, shears and moments for certain common cases of continuous girders, the diagrams of Figs. 304 to 312 inclusive have been prepared. Inspection of these diagrams shows that for a continuous girder of either two or three equal spans loaded with a uniform live load, w, per foot the maximum live moment

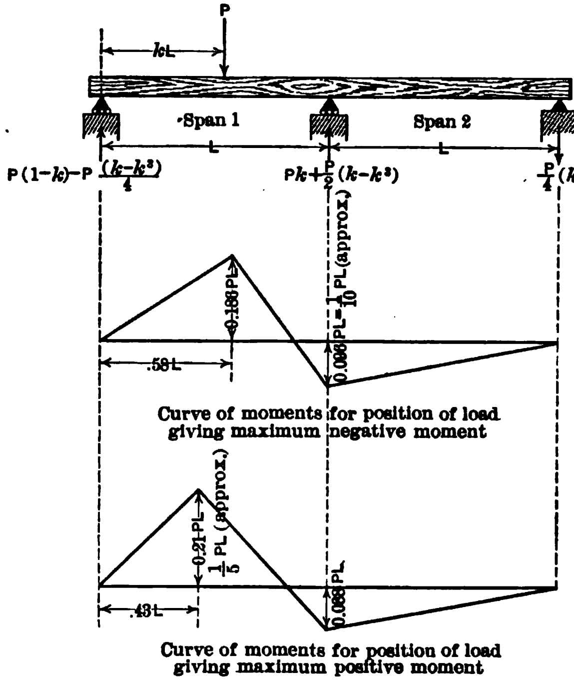
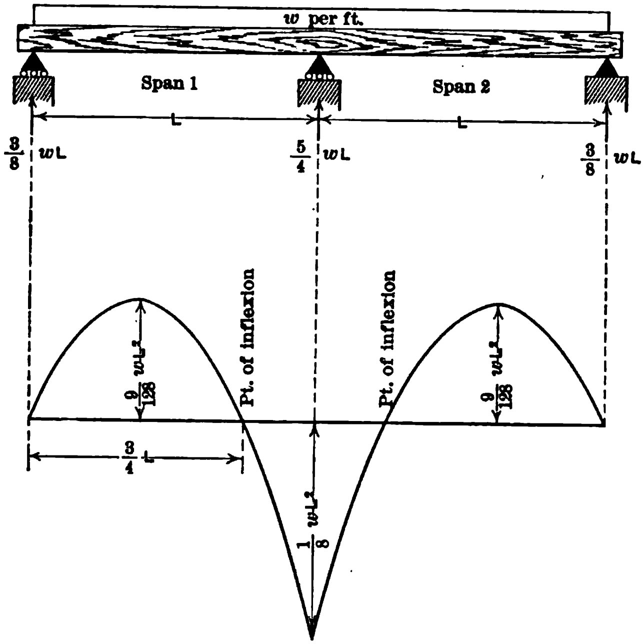
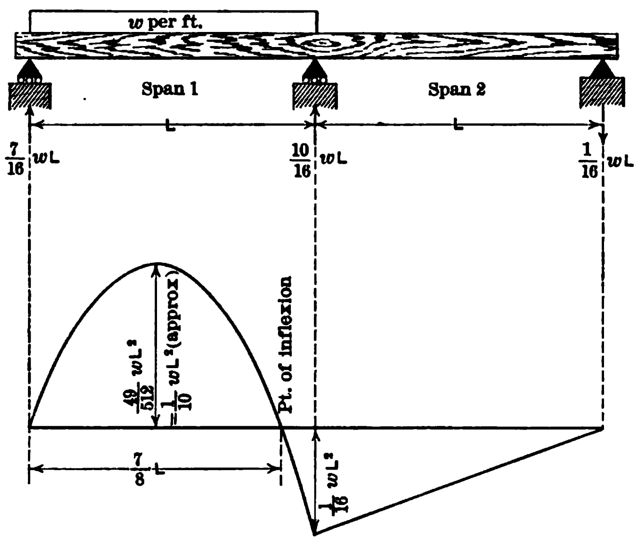


FIG. 304.—*Continuous Girder.* Reactions and Moments for a Single Concentrated Load.

occurs at a support and is negative, its value equalling, for the two-span girder, that of the positive live moment on an end-supported span, and being slightly less than this for the three-span girder. The maximum positive moment equals $\frac{1}{10} wL^2$ for both cases, or about three-quarters of the value it would have for an end-supported span.

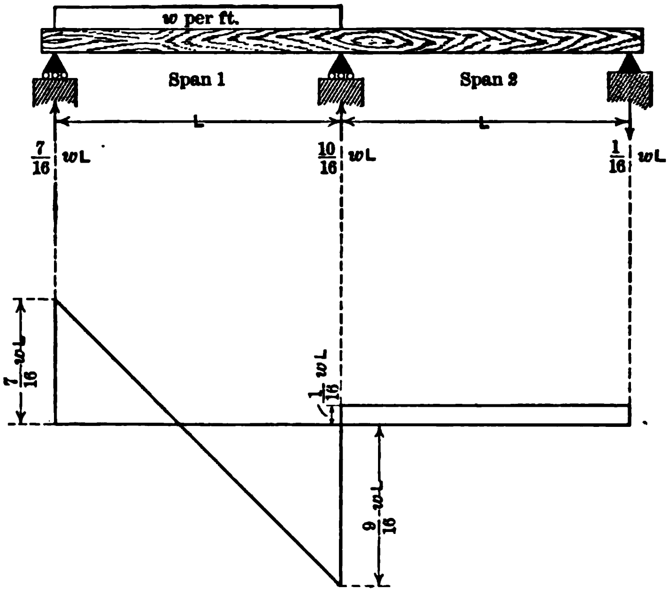


Curve of moments for uniform load = w per ft. over entire structure, This loading gives maximum negative moment.

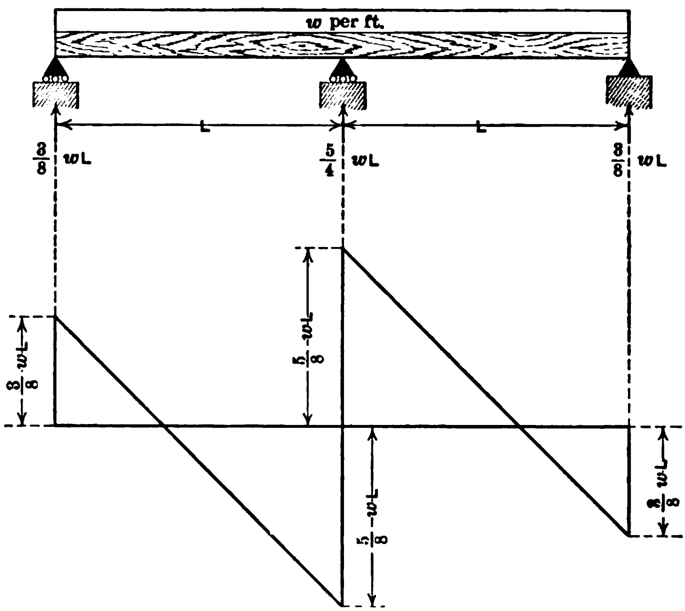


Curve of moments for uniform load = w per ft. on one span only, This loading gives maximum positive moment.

FIG. 305.—*Continuous Girder.* Curves of Moment for Uniform Load.



Curve of shears for uniform load $= w$ per ft. on one span only.



Curve of shears for uniform load $= w$ per ft. over both spans.
This loading gives maximum shear.

FIG. 306.—*Continuous Girder.* Curves of Shears for Uniform Load.

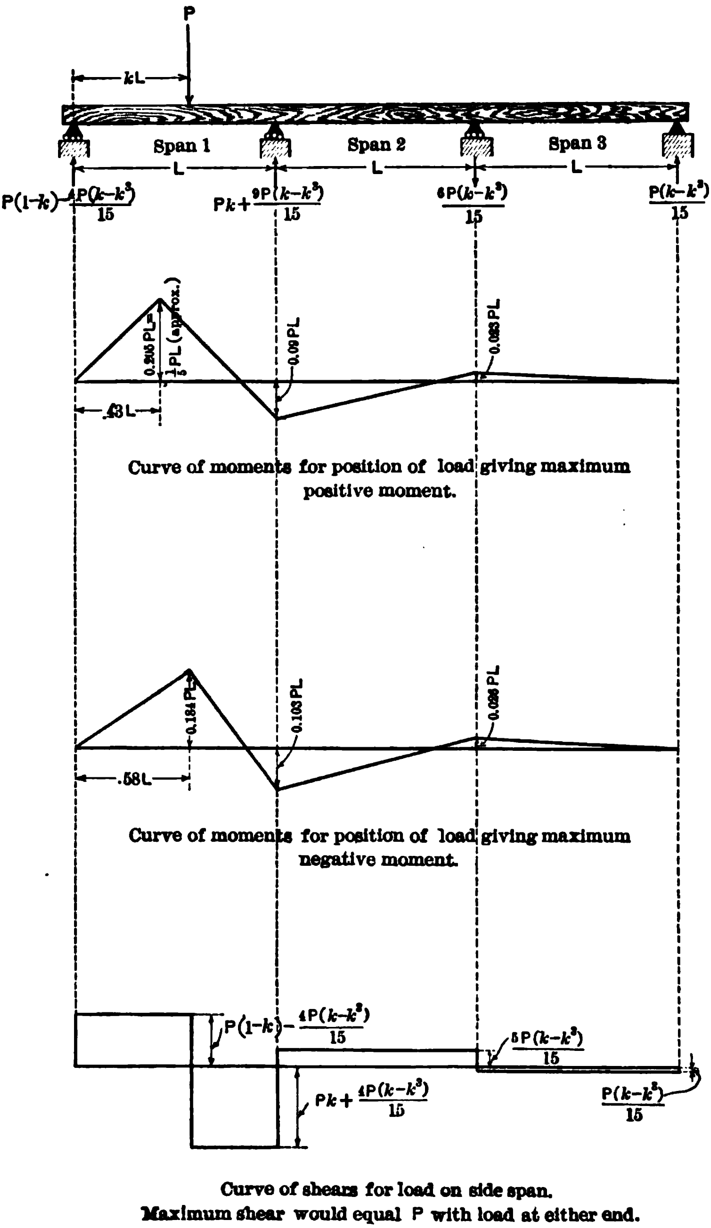


FIG. 307.—*Continuous Girder*. Curve of Shears and Moments. Single Concentrated Load.

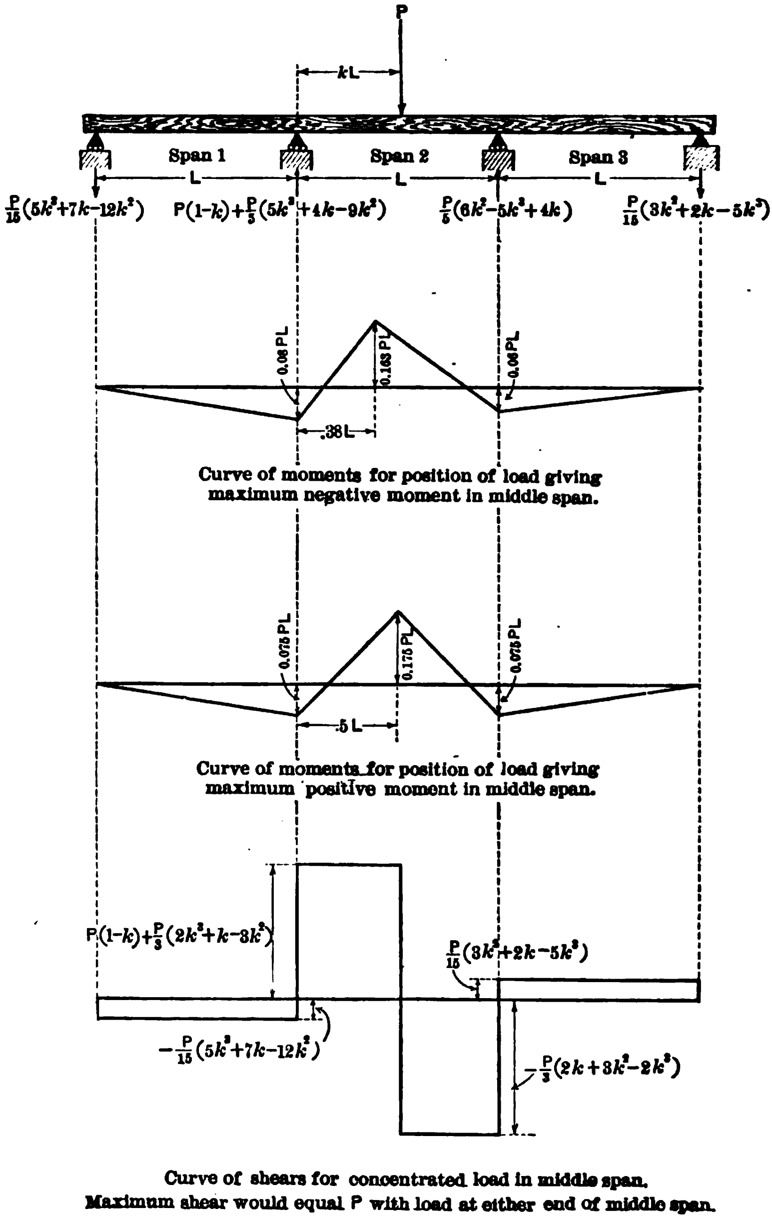
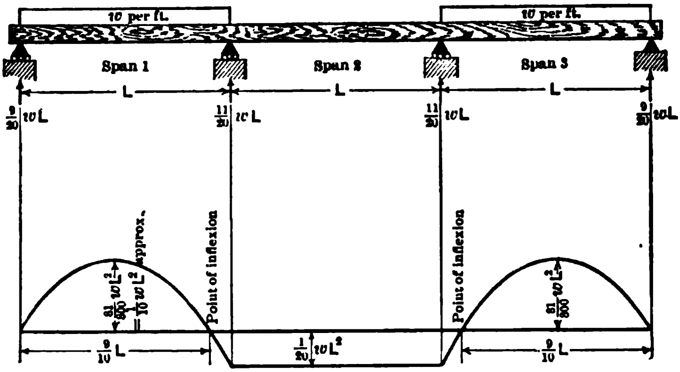
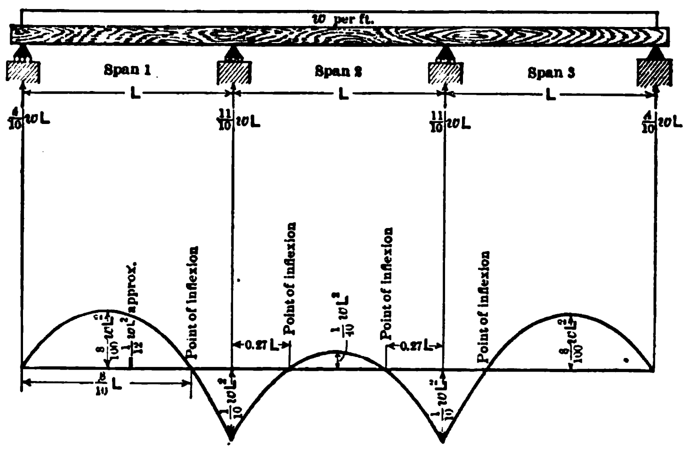


FIG. 308.—*Continuous Girder*. Curves of Moments and Shears. Single Concentrated Load.

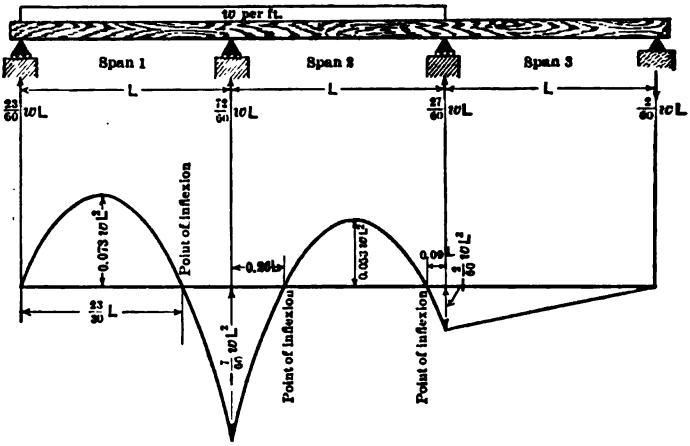


Curve of moments for live load = w per ft. on spans 1 and 3 only.
This loading gives maximum positive moment on spans 1 and 3.

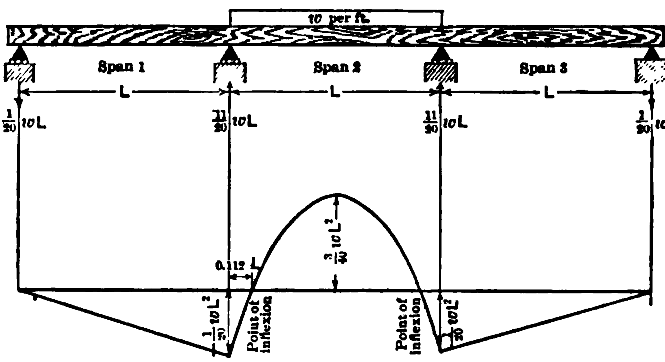


Curve of moments for live load = w per ft. over entire girder.

FIG. 309.—*Continuous Girder.* Curves of Moments for Uniform Load.

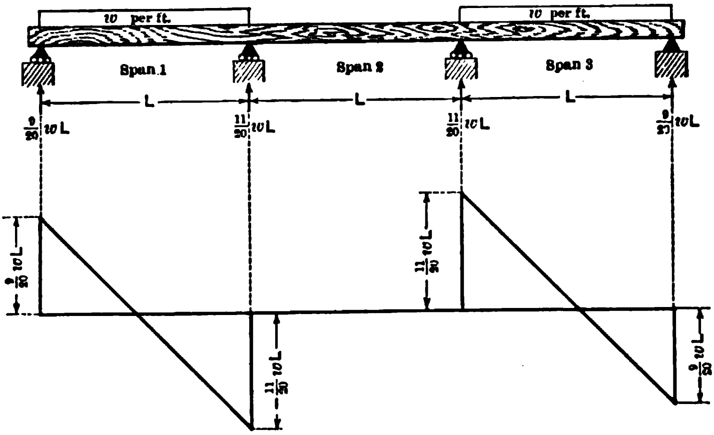


Curve of moments for live load $= w$ per ft. on spans 1 and 2.
This loading gives maximum negative moment on structure.

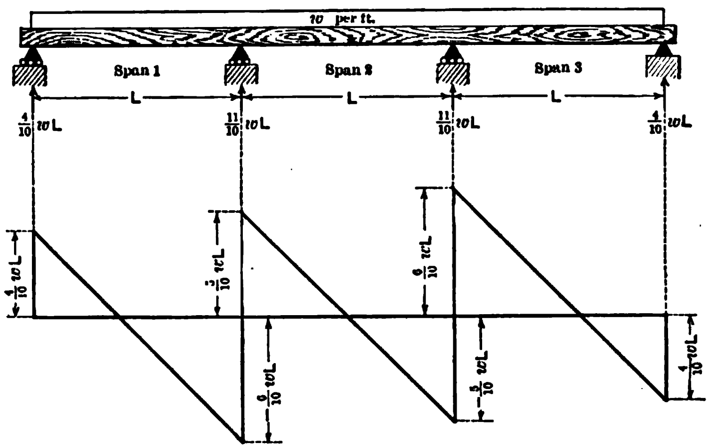


Curve of moments for live load $= w$ per ft. on span 2 only.
This loading gives maximum positive moment on span 2.

FIG. 310.—*Continuous Girder*. Curves of Moments for Uniform Loads.



Curve of shears for uniform load = w per ft. on spans 1 and 3 only.



Curve of shears for uniform load = w per ft. over entire structure.

FIG. 311.—*Continuous Girder.* Curves of Shears for Uniform Load.

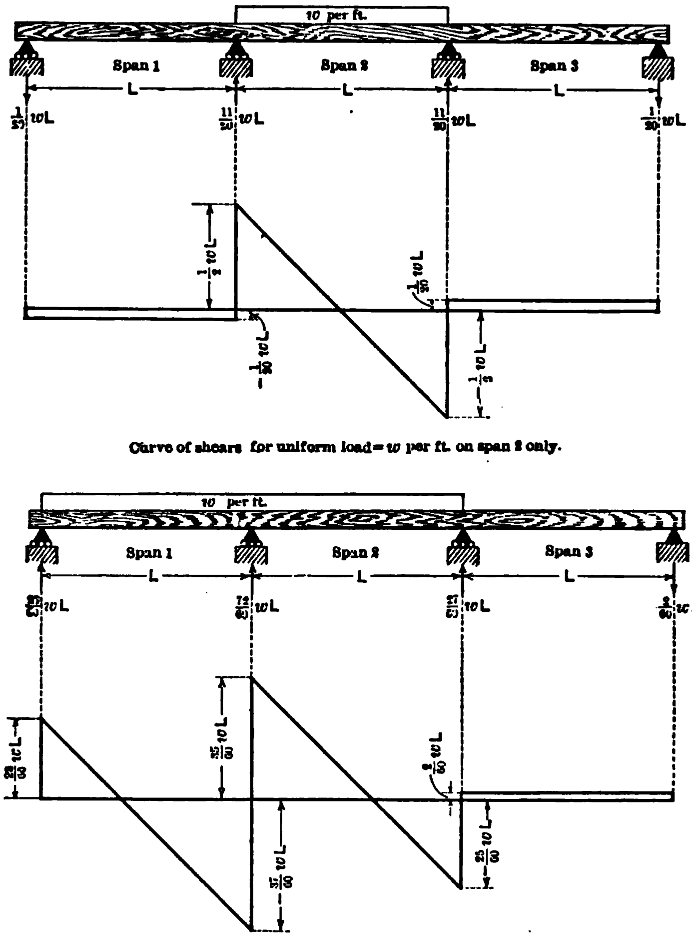
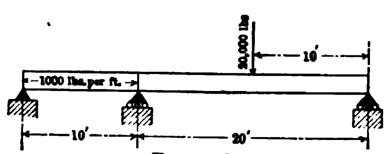


FIG. 312.—*Continuous Girder.* Curves of Shears for Uniform Load.

Problems

65. Determine reactions on this beam by application of the three-moment equation.



PROB. 65.

CHAPTER XVI

THEOREM OF LEAST WORK

187. Statement of Theorem. This theorem which is really nothing but a special application of the principle of the conservation of energy and as such needs no proof, was first brought to the attention of engineers by Castigliano (see bibliography at end of chapter). It may be stated as follows:

The internal work done in any stationary structure by a system of applied loads will be the least possible, consistent with equilibrium.

The theorem may be readily used to determine accurately the reactions or bar stresses in statically indeterminate structures provided the cross-sections of the various members have previously been approximately determined by other methods. The method of application to such cases is as follows:

Write an expression for the total work in the structure in terms of as many independent unknowns as would exist in excess of the number which can be determined by the equations of statics, differentiate the expression with respect to each of these unknowns, place each of these partial derivatives equal to zero and solve the resulting equations to determine the value of each unknown; e.g., in an end-supported planar structure having five unknown reaction components, any three of the components can be expressed in terms of the other two by application of the equations of equilibrium; it therefore follows that any two of these reaction components may be considered as independent variables in terms of which the work in the structure may be expressed and with respect to each of which the expression for work should be differentiated.

188. Expressions for Internal Work.

Case a. Bar subjected to direct axial stress only.

Let S = total direct axial stress in pounds applied to the bar.

A = area of cross-section of bar in square inches.

E = modulus of elasticity in pounds per square inch.

L = length of bar in feet.
W = total work in bar in foot-pounds due to force S.
δ = change in length of bar due to application of force P.

From mechanics it is known that the total internal work done in a bar by a gradually applied force equals one-half the product of the force by the change in length of the bar, and that practically all forces to which structure are subject may be considered to be gradually applied.

Also that
$$\delta = \frac{SL}{AE}$$

Hence,
$$W = \frac{1}{2}\frac{S^2L}{AE} \tag{45}$$

Case b. Bar subjected to bending only.

The expression for this case may be determined in the following manner referring to Fig. 286, Art. 177, but assuming the depth and width of the bar to be variable in order to obtain a general solution.

Let δ and f_2 be the same as in Art. 177.

w_0 = internal work in a prism of length dx, depth dy and width w with its center at a distance y from the neutral axis of the bar, this work being due to the application of outer forces.

I = moment of inertia in inches, about the neutral axis, with respect to flexure, of the cross-section of the bar, at any section such as fg.

M = bending moment in bar in inch-pounds at the same cross-section due to gradually applied loads.

W = total internal work in the bar in inch-pounds due to applied loads.

Then
$$\delta = \frac{f_2dx}{E},$$

and
$$w_0 = \frac{1}{2}(f_2bdy)\left(f_2\frac{dx}{E}\right)$$

but
$$f_2 = \frac{My}{I}$$

$$\therefore\ w_0 = \frac{1}{2}f_2^{\,2}\frac{bdydx}{E} = \frac{1}{2}\frac{M^2y^2}{I^2}\cdot\frac{bdydx}{E}$$

$$\therefore\ W=\int\frac{bM^2y^2dydx}{2EI^2}$$

But

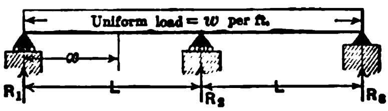
$$\int y^2bdy=I$$

$$\therefore\ W=\int\frac{M^2dx}{2EI} \tag{46}$$

The expression for the total work in a bar subjected to both direct stress and bending may evidently be obtained by combining equations (45) and (46).

It should be noted that in the expression for work due to bending the work resulting from the distortion due to shear is neglected. The effect of this, however, is not large and the stress due to it may be classed with secondary stresses which may usually be ignored in design.

189. Examples of Applications of Theorem of Least Work. The following example illustrates clearly the application of the theorem to a simple case.



Moment of inertia of beam = I.
Modulus of elasticity = E.

FIG. 313.

Problem. Determine by the theorem of least work the reactions on the beam shown in Fig. 313.

Solution. For this case there are four unknown reactions and three statical equations, hence only one unknown need be obtained by the theorem of least work. Let this unknown be taken as R_1. By statics,

$$R_3=R_1,$$
$$R_2=2wl-2R_1.$$

The internal work must now be expressed in terms of R_1. For the case under consideration the total work in the beam will be double that in the left span,

hence

$$W=\int_0^L\frac{M^2dx}{EI}$$

But

$$M=R_1x-\frac{wx^2}{2},$$

hence

$$W=\int_0^L\left(R_1x-\frac{wx^2}{2}\right)^2\frac{dx}{EI}=\frac{1}{EI}\left(\frac{R_1^2L^3}{3}-\frac{R_1wL^4}{4}+\frac{w^2L^5}{20}\right)$$

For a minimum value $\frac{dw}{dR_1}$ must equal zero, hence

$$\frac{2R_1L^3}{3}-\frac{wL^4}{4}=0 \quad \text{and} \quad R_1=\frac{3wL}{8}$$

190. Reactions in Continuous Trusses. The reactions in continuous trusses, made up of bar the cross-sections of which have been previously determined by approximate methods, such as the three-moment equation, may be readily determined with accuracy by the method of least work. To illustrate the method, equations for the typical continuous truss shown in Fig. 314 will be developed in this article. This truss is indeterminate to the second degree and the two central reactions R_b and R_c will be taken as the independent variables entering into the expression for work.

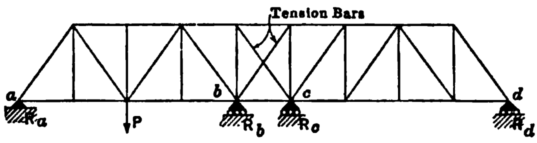


FIG. 314.

Now consider the truss as a simple end-supported structure subjected to the applied load P and the two unknown forces R_b and R_c, and under these conditions:

Let S_o = stress in any member due to applied load P.
S_b = stress in any member due to an upward force of unity acting at support b.
S_c = stress in any member due to an upward force of unity acting at support c.
Then R_bS_b = stress in any member due to the unknown upward reaction R_b.
and R_cS_c = stress in any member due to the unknown upward reaction R_c.

Hence, the actual stress in any member of the truss when considered as continuous $=S=S_o+R_bS_b+R_cS_c$.

The work in each bar $=\frac{S^2L}{2AE}$, hence the work in the entire truss $=\Sigma\frac{S^2L}{2AE}$.

Substituting for S its value as given in the previous equation, we obtain

$$W=\Sigma\frac{(S_o+R_bS_b+R_cS_c)^2L}{2AE}$$

hence
$$\frac{dW}{dR_b}=\Sigma(S_o+R_bS_b+R_cS_c)\frac{S_bL}{AE}=0$$ [1]

and
$$\frac{dW}{dR_c}=\Sigma(S_o+R_bS_b+R_cS_c)\frac{S_cL}{AE}=0$$

These equations may be expressed thus:

$$\Sigma\frac{S_oS_bL}{AE}+R_b\Sigma\frac{S_b^2L}{AE}+R_c\Sigma\frac{S_bS_cL}{AE}=0 \tag{47}$$

and
$$\Sigma\frac{S_oS_cL}{AE}+R_b\Sigma\frac{S_bS_cL}{AE}+R_c\Sigma\frac{S_c^2L}{AE}=0 \tag{48}$$

The solution of these equations gives the following values for the two unknowns:

$$R_b=\frac{\left(\Sigma\frac{S_oS_bL}{AE}\right)\left(\Sigma\frac{S_cL}{AE}\right)-\left(\Sigma\frac{S_oS_cL}{AE}\right)\left(\Sigma\frac{S_bS_cL}{AE}\right)}{\left(\Sigma\frac{S_bS_cL}{AE}\right)^2-\left(\Sigma\frac{S_c^2L}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)} \tag{49}$$

$$R_c=\frac{\left(\Sigma\frac{S_oS_cL}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)-\left(\Sigma\frac{S_oS_bL}{AE}\right)\left(\Sigma\frac{S_bS_cL}{AE}\right)}{\left(\Sigma\frac{S_bS_cL}{AE}\right)^2-\left(\Sigma\frac{S_c^2L}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)} \tag{50}$$

E is usually constant and may be cancelled from the previous equations giving the following equations:

$$R_b=\frac{\left(\Sigma\frac{S_oS_bL}{A}\right)\left(\Sigma\frac{S_c^2L}{A}\right)-\left(\Sigma\frac{S_oS_cL}{A}\right)\left(\Sigma\frac{S_bS_cL}{A}\right)}{\left(\Sigma\frac{S_bS_cL}{A}\right)^2-\left(\Sigma\frac{S_c^2L}{A}\right)\left(\Sigma\frac{S_b^2L}{A}\right)} \tag{51}$$

$$R_c=\frac{\left(\Sigma\frac{S_oS_cL}{A}\right)\left(\Sigma\frac{S_b^2L}{A}\right)-\left(\Sigma\frac{S_oS_bL}{A}\right)\left(\Sigma\frac{S_bS_cL}{A}\right)}{\left(\Sigma\frac{S_bS_cL}{A}\right)^2-\left(\Sigma\frac{S_c^2L}{A}\right)\left(\Sigma\frac{S_b^2L}{A}\right)} \tag{52}$$

[1] Note that in problems of this character the work is ordinarily considerably simplified by differentiating before performing the summation or integration.

The denominators of the two reaction equations (51) and (52) are alike and are independent of the position of the applied load. They also contain factors which occur in the numerators and are also independent of the applied load; hence these equations may be written in the following simple form.

$$R_b = \frac{m\Sigma\frac{S_oS_bL}{A} - n\Sigma\frac{S_oS_cL}{A}}{n^2 - mp} \tag{53}$$

$$R_c = \frac{p\Sigma\frac{S_oS_cL}{A} - n\Sigma\frac{S_oS_bL}{A}}{n^2 - mp} \tag{54}$$

in which n, m and p are constants for any particular structure, their values being as follows:

$$n = \Sigma\frac{S_bS_cL}{A}, \qquad m = \Sigma\frac{S_c^2L}{A}, \qquad p = \Sigma\frac{S_b^2L}{A}$$

It should be particularly noted that bars in which both S_b and S_c are zero may be entirely ignored in applying these equations. This evidently includes each bar the stress in which may be determined by statics.

If there are more than four supports, additional equations may be obtained in a similar manner. For example, if the number of unknown reaction components is $n+2$ the n equations necessary for solution may be derived from the equation of least work by expressing the stress in each member in terms of the n unknowns and differentiating n times, placing each derivative equal to zero.

If there is but one intermediate support the equation for R_b may be obtained by placing the last term in equation (47) equal to zero and solving the resulting expression giving the following result:

$$R_b = -\frac{\Sigma\frac{S_oS_bL}{AE}}{\Sigma\frac{S_o^2L}{AE}} \tag{55}$$

which for a constant value of E gives

$$R_b = -\frac{\Sigma\frac{S_oS_bL}{A}}{\Sigma\frac{S_b^2L}{A}} \tag{56}$$

As before, the denominator is a constant which is independent of the applied load and any bar in which S_b equals zero may be ignored in the computation; this includes each bar the stress in which may be computed by statics.

The equations deduced by this method give more accurate values than the three-moment equation inasmuch as the latter assumes a constant moment of inertia while the method of least work takes due account of the variation in the moment of inertia. On the other hand, the application of this method requires the pre-determination of the areas of all the members which cannot be done until the reactions are at least approximately determined.

A proper mode of procedure for such trusses is, therefore, to make a preliminary design based upon the determination of the reactions by the application of the three-moment equation, and then to check this by the determination of the reactions by the more accurate method of least work.

For the application of either method, a table should be prepared in which the various constants for each bar should be placed in vertical columns so that the summations may be obtained by adding the columns. Such a table should have the following headings for the case where E is constant and where there are two independent variables.

Bar	L Ft.	A Sq. in.	S_a Lbs.	S_b Lbs.	S_c Lbs.	$\frac{L}{A}$	$\frac{S_aS_bL}{A}$	$\frac{S_aS_cL}{A}$	$\frac{S_bS_cL}{A} = n$	$\frac{S_b^2L}{A} = p$	$\frac{S_c^2L}{A} = m$
U_0L_1......		Note	that	bars	in w	hic	h both	S_b	and	S_c are	zero,
U_1L_2......		may	be o	mitt	ed.						
etc.											
Summations											

In obtaining the summations, due attention should be given to the signs of the various terms; e.g., if for any bar S_a is tension and S_b is compression, the value of the product is negative.

To determine the maximum value of any one of the reactions or bar stresses influence lines may be drawn or influence tables constructed from which the position of the load for maximum values may readily be determined.

The examples which follow the text of this article illustrate

the application of the above table to the solution of an actual case.

191. Stresses in Trusses with Redundant Members. The method used for reactions may be applied equally well to trusses which are statically indeterminate with respect to the inner forces (i.e., to trusses with redundant members) by substituting for the unknown reactions the unknown stresses in the redundant

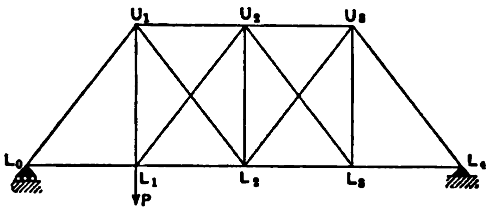


FIG. 315.

members. The equations for such cases may be derived as follows:

Consider the truss shown in Fig. 315 which has two redundant members. Consider these two redundant members to be U_1L_2 and L_2U_3, although any other two superfluous members could be used equally well.

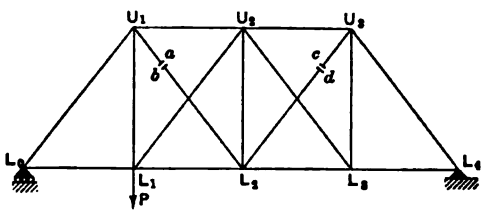


FIG. 316.

Now consider the same truss with bars U_1L_2 and L_2U_3 cut, such a truss being shown in Fig. 316, the cut ends a and b and c and d of the respective bars being an infinitesimal distance apart. For this latter truss

Let S_o = stress in any bar due to applied load P.

S_b = stress in any bar due to two forces of unity applied simultaneously at a and b in direction U_1L_2 and acting toward each other.

S_c = stress in any bar due to two forces of unity applied simultaneously at c and d in direction L_2U_3 and acting toward each other.

For the truss in Fig. 315 (the actual truss)

Let X = stress (assumed as tension) in redundant member U_1L_2.
Y = stress (assumed as tension) in redundant member L_2U_3.
S = actual stress in any member.
W = total work in structure.

Then $$S=S_o+S_bX+S_cY \quad . \quad . \quad . \quad . \qquad (57)$$

hence $$W=\Sigma\frac{(S_o+S_bX+S_cY)^2L}{2AE}$$

In these summations the work in the cut bars should be included with the other bars.

Differentiating the previous equations with respect to the two variables X and Y gives the following results:

$$\frac{dW}{dX}=\Sigma\left(\frac{S_o+S_bX+S_cY}{AE}\right)S_bL=0$$

$$\frac{dW}{dY}=\Sigma\left(\frac{S_o+S_bX+S_cY}{AE}\right)S_cL=0$$

Whence $$\Sigma\frac{S_oS_bL}{AE}+X\Sigma\frac{S_b^2L}{AE}+Y\Sigma\frac{S_bS_cL}{AE}=0 \quad . \quad . \quad . \quad (58)$$

and $$\Sigma\frac{S_oS_cL}{AE}+X\Sigma\frac{S_bS_cL}{AE}+Y\Sigma\frac{S_c^2L}{AE}=0 \quad . \quad . \quad . \quad (59)$$

The only difference in form between these equations and equations (47) and (48) lies in the substitution of X for R_b and Y for R_c, hence the values for X and Y obtained by solving these equations will be identical in form with the values for R_b and R_c as given by equations (49) and (50), viz.,

$$X=\frac{\left(\Sigma\frac{S_oS_bL}{AE}\right)\left(\Sigma\frac{S_c^2L}{AE}\right)-\left(\Sigma\frac{S_oS_cL}{AE}\right)\left(\Sigma\frac{S_bS_cL}{AE}\right)}{\left(\Sigma\frac{S_bS_cL}{AE}\right)^2-\left(\Sigma\frac{S_c^2L}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)} \qquad (60)$$

$$Y=\frac{\left(\Sigma\frac{S_oS_cL}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)-\left(\Sigma\frac{S_oS_bL}{AE}\right)\left(\Sigma\frac{S_bS_cL}{AE}\right)}{\left(\Sigma\frac{S_bS_cL}{AE}\right)^2-\left(\Sigma\frac{S_c^2L}{AE}\right)\left(\Sigma\frac{S_b^2L}{AE}\right)} \qquad (61)$$

If the value of E is constant in all cases it may and should be cancelled for all terms, thus reducing greatly the labor of computation.

These equations may be written in the same form as equations (53) and (54), E being assumed as constant, viz.,

$$X = \frac{m\Sigma\frac{S_oS_bL}{A} - n\Sigma\frac{S_oS_cL}{A}}{n^2 - mp} \quad . \quad . \quad . \quad . \quad (62)$$

$$Y = \frac{p\Sigma\frac{S_oS_cL}{A} - n\Sigma\frac{S_oS_bL}{A}}{n^2 - mp} \quad . \quad . \quad . \quad . \quad (63)$$

In which n, m and p have the values previously given.

The application of the foregoing formulas should be made by means of a table as previously shown. Bars in which both S_b and $S_c = 0$ should be omitted; this includes in the truss shown by Fig. 315 all bars such as L_0U_1, L_0L_1, etc., the stresses in which can be computed by statics.

To determine the position of the live load for maximum stress in any member, influence lines or influence tables may be used.

Equations for a truss with more than two redundant members may be derived in a similar manner as previously suggested in the article dealing with reactions.

For the case of a truss with one redundant member, the value of the stress in this member may be obtained by placing $Y = 0$ in equation (58) giving the following result:

$$X = -\frac{\Sigma\frac{S_oS_bL}{AE}}{\Sigma\frac{S_b^2L}{AE}} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

which takes the following form for a structure in which E is constant.

$$X = -\frac{\Sigma\frac{S_ob_bL}{A}}{\Sigma\frac{S_b^2L}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

The illustrative problems which follow clearly show the method of application of the equations.

Problem. Compute the stresses in the redundant members of the truss, shown by Fig. 317, due to the loads shown. Gross areas of various members are given in the table accompanying the solution. The modulus of elasticity is assumed to be constant throughout.

Solution. This truss is evidently indeterminate to the second degree; bars L_1U_2 and U_2L_3 will be considered the redundant members, although any two bars except L_0U_1, L_0L_1, U_3L_4, and L_3L_4 might be used. S_b is the stress caused by a force of unity in bar L_1U_2 and S_c the stress caused by a force of unity in bar U_2L_3.

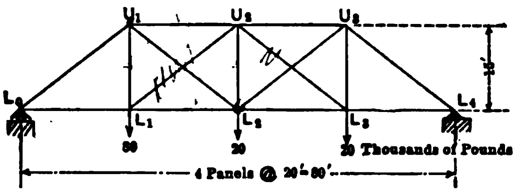


FIG. 317.

The following table contains all the necessary values and the equations beneath it give the final results.

Bar.	L in Ft.	A in Sq. In.	S_o 1000 lb. units	S_b	S_c	$\frac{S_oS_bL}{A}$	$\frac{S_oS_cL}{A}$	$\frac{S_bS_cL}{A}$	$\frac{S_b^2L}{A}$	$\frac{S_c^2L}{A}$
U_1U_2	20	12	− 93.3	−0.8	0.0	+124.4	0.0	0.0	+1.1	0.0
U_2U_3	20	12	− 93.3	0.0	−0.8	0.0	+124.4	0.0	0.0	+1.1
L_1L_2	20	8	+100.0	−0.8	0.0	−200.0	0.0	0.0	+1.6	0.0
L_2L_3	20	8	+ 60.0	0.0	−0.8	0.0	−120.0	0.0	0.0	+1.6
U_1L_2	25	4	− 8.3	+1.0	0.0	− 52.1	0.0	0.0	+6.2	0.0
L_2U_3	25	4	+ 41.7	0.0	+1.0	0.0	+260.4	0.0	0.0	+6.2
U_1L_1	15	6	+ 80.0	−0.6	0.0	−120.0	0.0	0.0	+0.9	0.0
U_3L_3	15	6	+ 20.0	0.0	−0.6	0.0	− 30.0	0.0	0.0	+0.9
L_1U_2	25	4	0.0	+1.0	0.0	0.0	0.0	0.0	+6.2	0.0
U_2L_3	25	4	0.0	0.0	+1.0	0.0	0.0	0.0	0.0	+6.2
U_2L_2	15	4	0.0	−0.6	−0.6	0.0	0.0	+1.3	+1.4	+1.4
Summations						−247.7	+234.8	+1.3	+17.4	+17.4

$$X = \text{stress in } L_1U_2 = \frac{-247.7\times17.4-234.8\times1.3}{1.3^2-17.4^2}$$

$$= \frac{-4310-305.2}{1.7-302.7} = +15.3$$

$$Y = \text{stress in } U_2L_3 = \frac{+234.8\times17.4+247.7\times1.3}{-301}$$

$$= \frac{4085.5+322.0}{-301} = -\frac{4407.5}{301} = -14.6$$

With X and Y determined, stresses in all other bars may be readily computed by applying equation (57).

Problem. Compute stress in redundant member U_3U_4 of the truss shown by Fig. 318, due to load shown. Areas of all members are given in table accompanying the solution and are selected arbitrarily as factors of bar lengths to save numerical work, the truss being a hypothetical one. E is constant for all members.

Solution. Since the truss has but one redundant member equation (65) is applicable. The following table gives all necessary values; the determination of the stress follows the table.

Bar.	L in Ft.	A in Sq. In. Gross.	S_o* in 1000 lb. units	S_b	$\frac{L}{A}$	$\frac{S_oS_bL}{A}$*	$\frac{S_b^2L}{A}$
L_0U_1........	39	13.0	−26.0	+0.69	3.0	− 53.8	+ 1.43
U_2L_3........	39	13.0	−52.0	−0.17	3.0	− 26.5	+ 0.09
L_0L_1........	25	10.0	+16.7	−0.44	2.5	− 18.5	+ 4.84
L_1L_2........	25	10.0	+33.3	−0.89	2.5	− 74.2	+ 1.98
L_2L_3........	25	10.0	+33.3	−0.89	2.5	− 74.2	+ 1.98
U_1L_1........	30	10.0	+20.0	−0.53	3.0	− 31.8	+ 0.85
L_1U_2........	39	13.0	−26.0	+0.69	3.0	− 53.8	+ 1.43
U_1U_2........	25	12.5	−33.3	+0.44	2.0	− 29.3	+ 0.39
U_2U_3........	27	9.0		+1.08	3.0		+ 3.50
U_3L_3........	40	8.0		−0.40	5.0		+ 0.80
Σ for half	truss	neglect	ing bars	U_3U_4 and	L_3L_4.	=	+17.29
Σ for entire	truss	neglect	ing bars	U_3U_4	and	L_3L_4. =	+34.58
L_3L_4........	20	8.0		−1.00	2.5		+ 2.50
U_3U_4........	20	10.0		+1.00	2.0		+ 2.00
Σ for entire	truss.					−362.1	+39.08

*S_o equals zero for all bars except those for which numerical values are given; hence, $\frac{S_oS_bL}{A}$ = zero for same bars.

$$X = \text{stress in } U_3U_4 = +\frac{362.1}{39.08} = +9.3$$

Stresses in other bars can be found by application of equation (57) with last term omitted.

$$\text{Uplift at } L_7 = 9.3 \times \frac{40}{75} = 5.00$$

192. Influence Lines and Tables for Indeterminate Structures. The construction of influence lines or tables for reactions or bar stresses for indeterminate trusses is comparatively simple

since S_o is the only variable for any particular truss. Considering for example the truss computed in the preceding illustrative problem it is evident that in obtaining data for an influence line, S_o should first be computed for a load of unity at any convenient point, and the products in the last column but one of the table computed for that value. If we now let S'_o = bar stress due to load unity at any other panel point, the value of the stress X in U_3U_4 for this latter position can be determined by multiplying each separate value in the previously mentioned column by $\frac{S'_o}{S_o}$, obtaining a new summation for that column, and using the other summations without modification.

It may be interesting to note that since $\frac{L}{A}$ appears in general in each term of numerator and denominator of the fraction giving the stress in bar U_3U_4 of such a truss, a slight approximation in the value of A, would not affect the final result greatly. In fact, in the illustrative example just given, it would make no material difference in the final result if $\frac{L}{A}$ should be taken as unity throughout.

193. Stresses in Indeterminate Structure due to Changes of Temperature. To determine bar stresses or reactions due to

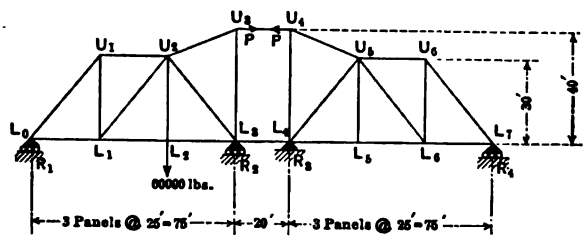


FIG. 318.

changes of temperature in any or all members of a statically indeterminate structure the equation previously deduced may be used, the values for S_o being not those due to an applied load but instead the forces required to change the lengths of the various bars an amount equal to the change due to variations in temperature. For example, if for the truss shown by Fig. 318 it is desired to compute the stress in bar U_3U_4 due to an increase in

temperature of the entire top chord, it is necessary to use for S_o for each separate bar of the top chord the axial force which would cause a change in length equal to that due to the change in temperature.

In considering the effect of a change of temperature in statically indeterminate trusses it should be noted that a uniform change in temperature of the entire truss, if the truss reactions are vertical, will not cause stresses in any member but that a similar change will cause material stresses in many if not most of the bars of a structure such as an arch which is rigidly restrained against horizontal movement. It should also be observed that a change in temperature of some but not all bars of a truss with vertical reactions will also stress many members.

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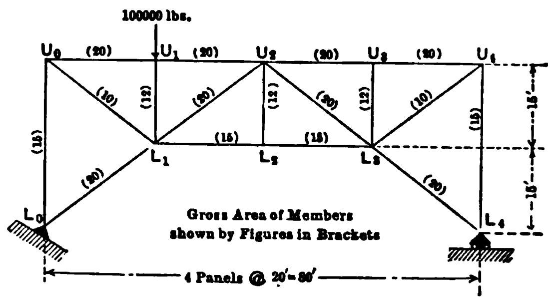
PROBLEMS

66. Compute by method of least work the horizontal reaction component on this two-hinged arch due to load shown.



PROB. 66.

67. Compute by method of least work the horizontal reaction component on same structure due to an increase of 60° F. in temperature of all members.

CHAPTER XVII

SWING BRIDGES

194. Movable Bridges—General. The term movable bridge is applied to all bridges which may be temporarily changed in position to allow passage of vessels in the streams which they span. Such bridges may be classified in the following six general divisions:

1. Swing Bridges.
   a. Center supported.
   b. Rim supported.
2. Bascule Bridges.
   (Bridges which revolve vertically about one end.)
3. Direct Lift Bridges.
   (Bridges which are lifted bodily.)
4. Retractile Bridges.
   (Bridges which are moved in a horizontal plane.)
5. Ferry or Transporter Bridges.
   (Birdges supporting a movable car which transports vehicles or passengers.)
6. Pontoon Bridges.
   (Bridges built on boats.)

It is the purpose of this chapter to deal merely with the stresses in swing bridge trusses; no attention will be given to the other types except for the brief reference which follows to stresses in bascule bridge trusses. The design of the machinery for movable bridges will not be considered as to do this thoroughly would require a large amount of space.

195. Stresses in Bascule Bridges. Bascule bridges may consist of one leaf which when closed acts as a simple end-supported span so far as the live load is concerned, or they may be made up of two leaves which are connected at the centre, when the bridge is closed, by a lock which can transmit shear but not bending moment, thus giving a structure which acts under live load like a

three-hinged arch. Since counterweights must be used to balance the dead weight, there will be no dead reaction at the end away from the centre of rotation and the maximum dead stresses may occur with the bridge partially or fully opened or with the bridge closed. The following rules for determining the maximum dead stress in any member may be applied:

Let S_h = dead stress, bridge closed.
S_v = dead stress, bridge standing at 90° to closed position.
S = maximum dead stress.
θ = angle of opening, between closed bridge and position of bridge at which maximum stress occurs in the given member.

a. If S_h and S_v are of the same character, the maximum stress and the angle at which it occurs will be given by the following expressions:

$$S=\sqrt{S_h^2+S_v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

$$\tan\theta=\frac{S_v}{S_h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

b. If S_h and S_v are unlike in character, the maximum stress equals the larger of the two values S_h and S_v. (See article entitled "Maximum Stresses in Bascule Bridges," by W. W. Pagon, Trans., Am. Soc. C. E., Vol. LXXVI, p. 73.)

196. Types of Girders and Trusses for Swing Bridges. Main girders and trusses of swing bridges are statically determined with respect to the outer forces when the bridge is open, but are usually statically undetermined for the condition which exists when the bridge is closed and subjected to live load. The girders of plate girder bridges under the latter condition generally act as continuous girders supported at three points. Trusses may, however, be constructed according to any one of the following types:

a. Continuous—supported at three points.
b. Continuous—supported at four points.
c. Partially continuous—supported at four points.
d. Discontinuous—supported at four points.

The difference between types *b* and *c* lies in the fact that the latter type is so constructed that moment but not shear can be transmitted across the central panel, thus producing a condition similar to that described for cantilever trusses in Art. 125.

Type *c* is illustrated by Fig. 318 and is the type generally employed in the United States for trusses supported at four points. Its principal advantage over type *b*, also supported at four points, is that it cannot be loaded in such a manner as to cause uplift at either of the central points of support where provision for resisting uplift cannot readily be made. Type *d* can be readily constructed by providing adjustment in the top chord eyebars at the connection to the tower posts by means of oblong holes in the eyebars. Such trusses have been used for several important bridges in the United States, but have proven uneconomical to operate because of the expense required to raise the ends when closing the bridge.

197. Points of Support for Swing Bridges. The points of support of the main girders or trusses are usually upon cross-girders. In the class *a* type one such girder is required, the cross-girder itself being supported on a pivot resting on the central pier. In classes *b*, *c* and *d*, two cross-girders are needed, these girders being supported either upon a circular girder or drum, or upon other girders so arranged as to distribute the reactions more uniformly over the drum than would otherwise be possible. The circular girder or drum is itself supported upon a ring of conical rollers running upon a drum supported upon the central pier.

198. Swing Bridge Girders and Trusses—Equations for Reactions. If the girders or trusses correspond to the conditions assumed in developing the three-moment equation, that is, if their moments of inertia and moduli of elasticity are constant throughout their length, the reactions may be accurately computed for Cases *a* and *b* by the use of the three-moment equation, the labor of calculation being much simplified by the use of the special equations and tables of this chapter. For Case *c* special formulas are derived in the following article by the application of the method of least work and tabular values are given at the end of the chapter.

199. Determination of Reactions on a Partially Continuous Girder. For a truss similar to that shown in Fig. 318 the method of the previous article may also be applied. For this case there are five unknown reactions. All of these, however, can be expressed in terms of R_1 by applying the equation of statics accompanied by the equation of condition, viz., that the shear in the centre panel equals zero. The resulting values of the reactions will be found to be as shown in Fig. 319.

The internal work in this girder is given by the following equations, it being assumed that E and I are constant throughout:

$$W=\frac{1}{2EI}\int_0^{kL_1}(R_1x)^2dx+\frac{1}{2EI}\int_0^{L_1(1-k)}[R_1kL_1+(R_1-P)x]^2dx$$
$$+\frac{1}{2EI}\int_0^{L_2}\left[\left(\frac{R_1L_1+PL_1(k-1)}{L_3}\right)(L_3)\right]^2dx$$
$$+\frac{1}{2EI}\int_0^{L_3}\left(\frac{R_1L_1+PL_1(k-1)}{L_3}\right)^2x^2dx.$$

Determining the value of $\frac{dW}{dR_1}$ from above equation and equating to zero gives the following value for R_1:

$$R_1=P(1-k)\frac{\frac{L_1}{6}(1-k)(k+2)+L_2+\frac{L_3}{3}}{\frac{L_1}{3}+L_2+\frac{L_3}{3}}.$$

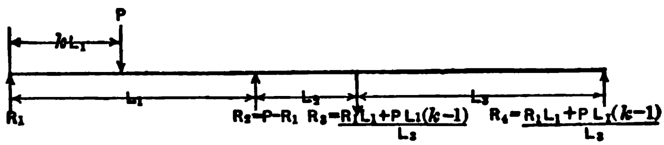


FIG. 319.

For the special case in which $L_1=L_3=nL_2$ where n equals the number of panels in each arm of the truss, we may substitute L for L_1 and L_3 and $\frac{L}{n}$ for L_2 giving the following expressions for the reactions, the signs indicating the actual direction of the reactions as compared with the directions assumed in Fig. 319.

$$R_1=P(1-k)-\frac{Pn(k-k^3)}{4n+6},$$

$$R_2=Pk+\frac{Pn(k-k^3)}{4n+6},$$

$$R_3=-\frac{Pn(k-k^3)}{4n+6},$$

$$R_4=-\frac{Pn(k-k^3)}{4n+6}$$

200. Influence of End Supports upon Swing Bridge Reactions. The continuous and partially continuous girders hitherto con-

sidered have been assumed to be level and supported on level supports. It is evident that if this condition exists for a closed swing bridge, the trusses will deflect at the ends below the level of the centre supports when the bridge is opened and hence when the bridge is again closed will have to be raised to reach their original level, the force required to accomplish this equalling the dead reactions which would exist when the truss is closed. If the ends be not raised there will be no dead reactions at the ends when closed, the dead stresses being the same as when the bridge is open. If this latter condition exists, however, a partial live loading which would tend to produce negative reactions at the ends would cause the ends to rise unless latched down, a serious objection, especially for a double-track railroad bridge. It is common, therefore, in the case of railroad swing bridges to raise the ends when the bridge is closed, this being accomplished by means of levers, toggle joints, or hydraulic jacks. In case the ends are raised sufficiently, the reactions for the closed bridge for both live and dead loads will be given by the formulas already deduced. If the ends are latched down and not raised, the end dead reactions will be zero and the live reactions will be given by the formulas of this chapter. If the ends are neither latched down nor raised, the end dead reactions will again be zero, and the live reactions will be those given by the formulas, provided none of the end reactions be negative and the negative live reaction at the centre pier does not exceed the positive dead reaction at that point. If the latter conditions are not fulfilled the structure becomes a girder supported at two points if of the class *a* type, and at three points if of types *b* and *c*, and the formulas are inapplicable. It should be added that the partially continuous truss has an advantage over the continuous girder upon four points of support, in having the negative reaction due to live loads occur at one end where it may be properly taken care of instead of at the centre support, where it might cause the drum to lift from the rollers.

201. Tables of Reactions for Continuous and Partially Continuous Girders Used for Swing Bridges. The actual stresses in girders used for swing bridges may be computed by the methods already given for simple girders and trusses, once the reactions are determined. It may be noted that influence tables or influence lines may be employed to advantage.

TABLE 1

REACTIONS FOR UNIT LOAD—GIRDER CONTINUOUS OVER TWO EQUAL SPANS

Moment of Inertia and Modulus of Elasticity Assumed to be Constant.

Positive signs indicate upward reactions.

Formulas used in deriving these values are determined by the three-moment equation and are as follows:

$$R_1=(1-k)-\left(\frac{k-k^3}{4}\right).$$

$$R_3=-\left(\frac{k-k^3}{4}\right).$$

$$R_2=1-(R_1+R_3).$$

k	R_1 +	R_2 +	R_3 −
	$n=10$		
1/10	0.8752	0.1495	0.0247
2/10	0.7520	0.2960	0.0480
3/10	0.6318	0.4365	0.0683
4/10	0.5160	0.5680	0.0840
5/10	0.4062	0.6875	0.0937
6/10	0.3040	0.7920	0.0960
7/10	0.2108	0.8785	0.0893
8/10	0.1280	0.9440	0.0720
9/10	0.0572	0.9855	0.0427
Total	3.8812	5.7375	0.6187
	$n=9$		
1/9	0.8615	0.1659	0.0274
2/9	0.7250	0.3278	0.0528
3/9	0.5926	0.4815	0.0741
4/9	0.4664	0.6228	0.0892
5/9	0.3484	0.7476	0.0960
6/9	0.2407	0.8519	0.0926
7/9	0.1454	0.9314	0.0768
8/9	0.0645	0.9821	0.0466
Total	3.4445	5.1110	0.5555
	$n=8$		
1/8	0.8442	0.1866	0.0305
2/8	0.6914	0.3672	0.0586
3/8	0.5444	0.5362	0.0806
4/8	0.4062	0.6875	0.0937
5/8	0.2798	0.8154	0.0952
6/8	0.1680	0.9140	0.0820
7/8	0.0737	0.9776	0.0513
Total	3.0077	4.4845	0.4919
	$n=7$		
1/7	0.8222	0.2128	0.0350
2/7	0.6487	0.4169	0.0656
3/7	0.4840	0.6035	0.0875
4/7	0.3324	0.7638	0.0962
5/7	0.1982	0.8893	0.0875
6/7	0.0860	0.9709	0.0569
Total	2.5715	3.8572	0.4287
	$n=6$		
1/6	0.7928	0.2477	0.0405
2/6	0.5926	0.4815	0.0741
3/6	0.4062	0.6875	0.0937
4/6	0.2407	0.8519	0.0926
5/6	0.1030	0.9606	0.0636
Total	2.1353	3.2292	0.3645
	$n=5$		
1/5	0.7520	0.2960	0.0480
2/5	0.5160	0.5680	0.0840
3/5	0.3040	0.7920	0.0960
4/5	0.1280	0.9440	0.0720
Total	1.7000	2.6000	0.3000
	$n=4$		
1/4	0.6914	0.3672	0.0586
2/4	0.4062	0.6875	0.0937
3/4	0.1680	0.9140	0.0820
Total	1.2656	1.9687	0.2343
	$n=3$		
1/3	0.5926	0.4815	0.0741
2/3	0.2407	0.8519	0.0926
Total	0.8333	1.3334	0.1667

TABLE 2

REACTIONS FOR UNIT LOADS—CONTINUOUS GIRDER WITH FOUR SUPPORTS AND EQUAL SIDE SPANS

Moment of Inertia and Modulus of Elasticity Assumed to be Constant.

Centre span $= \frac{1}{n}$ side span.

Positive signs indicate upward reactions.

Formulas used in deriving these values are determined by the three-moment equation and are as follows:

$$R_1 = (1-k) - \frac{(k-k^3)(2n)(n+1)}{4n^2+8n+3}. \qquad R_4 = \frac{(k-k^3)n}{4n^2+8n+3}.$$

$$R_2 = k + \frac{(k-k^3)(n)(2n^2+5n+2)}{4n^2+8n+3}. \qquad R_3 = -\frac{(k-k^3)n(2n^2+3n+1)}{4n^2+8n+3}.$$

k	$R_1(+)$	$R_2(+)$	$R_3(-)$	$R_4(+)$
		$n=10$		
1/10	0.8549	0.6166	0.4735	0.0021
2/10	0.7125	1.2018	0.9183	0.0040
3/10	0.5757	1.7243	1.3057	0.0056
4/10	0.4470	2.1531	1.6070	0.0069
5/10	0.3292	2.4567	1.7936	0.0078
6/10	0.2251	2.6036	1.8367	0.0079
7/10	0.1374	2.5628	1.7075	0.0074
8/10	0.0688	2.3027	1.3774	0.0060
9/10	0.0221	1.7922	0.8179	0.0035
Total	3.3727	17.4138	11.8376	0.0512
		$n=9$		
1/9	0.8394	0.6285	0.4703	0.0025
2/9	0.6825	1.2181	0.9054	0.0048
3/9	0.5330	1.7301	1.2697	0.0067
4/9	0.3946	2.1257	1.5284	0.0081
5/9	0.2712	2.3660	1.6459	0.0087
6/9	0.1662	2.4126	1.5871	0.0084
7/9	0.0836	2.2261	1.3166	0.0069
8/9	0.0269	1.7683	0.7995	0.0042
Total	2.9974	14.4754	9.5229	0.0503
		$n=8$		
1/8	0.8202	0.6433	0.4667	0.0031
2/8	0.6455	1.2368	0.8882	0.0058
3/8	0.4813	1.7318	1.2212	0.0080
4/8	0.3328	2.0790	1.4210	0.0093
5/8	0.2052	2.2287	1.4433	0.0094
6/8	0.1037	2.1316	1.2435	0.0081
7/8	0.0336	1.7387	0.7773	0.0051
Total	2.6223	11.7899	7.4612	0.0488
		$n=7$		
1/7	0.7956	0.6615	0.4609	0.0038
2/7	0.5991	1.2581	0.8643	0.0072
3/7	0.4177	1.7250	1.1524	0.0096
4/7	0.2596	1.9975	1.2678	0.0106
5/7	0.1320	2.0107	1.1524	0.0096
6/7	0.0430	1.7001	0.7492	0.0063
Total	2.2470	9.3529	5.6470	0.0471
		$n=6$		
1/6	0.7635	0.6852	0.4537	0.0050
2/6	0.5391	1.2814	0.8296	0.0091
3/6	0.3385	1.7000	1.0499	0.0115
4/6	0.1738	1.8518	1.0369	0.0114
5/6	0.0570	1.6480	0.7128	0.0078
Total	1.8719	7.1664	4.0829	0.0448
		$n=5$		
1/5	0.7194	0.7169	0.4430	0.0067
2/5	0.4590	1.3046	0.7754	0.0117
3/5	0.2389	1.6338	0.8861	0.0134
4/5	0.0792	1.5754	0.6646	0.0101
Total	1.4965	5.2307	2.7691	0.0419
		$n=4$		
1/4	0.6553	0.7612	0.4262	0.0095
2/4	0.3485	1.3182	0.6819	0.0151
3/4	0.1174	1.4660	0.5966	0.0133
Total	1.1212	3.5454	1.7047	0.0379
		$n=3$		
1/3	0.5538	0.8272	0.3951	0.0141
2/3	0.1922	1.2840	0.4938	0.1076
Total	0.7460	2.1112	0.8889	0.0317

TABLE 3

REACTIONS FOR UNIT LOADS—PARTIALLY CONTINUOUS GIRDER WITH FOUR SUPPORTS AND EQUAL SIDE SPANS

Shear in centre panel = 0

Moment of Inertia and Modulus of Elasticity Assumed to be Constant.

Centre span $= \frac{1}{n}$ side span.

Positive signs indicate upward reactions.

Formulas used in deriving these values are derived by the method of least work and are as follows:

$$R_1 = P(1-k) - \frac{Pn(k-k^3)}{4n+6} \qquad R_2 = Pk + \frac{Pn(k-k^3)}{4n+6}$$

$$R_3 = -R_4 = P(1-k) - R_1 = \frac{Pn(k-k^3)}{4n+6}$$

k	R_1 +	R_2 +	R_3 +	R_4 −
		$n=10$		
1/10	0.8785	0.1215	0.0215	0.0215
2/10	0.7583	0.2417	0.0417	0.0417
3/10	0.6406	0.3593	0.0593	0.0593
4/10	0.5269	0.4730	0.0730	0.0730
5/10	0.4185	0.5815	0.0815	0.0815
6/10	0.3165	0.6835	0.0835	0.0835
7/10	0.2224	0.7776	0.0776	0.0776
8/10	0.1374	0.8626	0.0626	0.0626
9/10	0.0628	0.9372	0.0372	0.0372
Total	3.9619	5.0379	0.5379	0.5379
		$n=9$		
1/9	0.8654	0.1346	0.0235	0.0235
2/9	0.7325	0.2675	0.0453	0.0453
3/9	0.6032	0.3968	0.0635	0.0635
4/9	0.4791	0.5209	0.0764	0.0764
5/9	0.3621	0.6379	0.0823	0.0823
6/9	0.2540	0.7460	0.0794	0.0794
7/9	0.1564	0.8436	0.0658	0.0658
8/9	0.0711	0.9289	0.0400	0.0400
Total	3.5238	4.4762	0.4762	0.4762
		$n=8$		
1/8	0.8491	0.1509	0.0259	0.0259
2/8	0.7007	0.2993	0.0493	0.0493
3/8	0.5571	0.4428	0.0678	0.0678
4/8	0.4210	0.5789	0.0789	0.0789
5/8	0.2948	0.7052	0.0802	0.0802
6/8	0.1809	0.8191	0.0691	0.0691
7/8	0.0818	0.9182	0.0432	0.0432
Total	3.0854	3.9144	0.4144	0.4144
		$n=7$		
1/7	0.8283	0.1717	0.0288	0.0288
2/7	0.6603	0.3397	0.0540	0.0540
3/7	0.4994	0.5006	0.0720	0.0720
4/7	0.3493	0.6507	0.0792	0.0792
5/7	0.2137	0.7863	0.0720	0.0720
6/7	0.0960	0.9040	0.0468	0.0468
Total	2.6470	3.3530	0.3528	0.3528
		$n=6$		
1/6	0.8009	0.1991	0.0324	0.0324
2/6	0.6074	0.3926	0.0593	0.0593
3/6	0.4250	0.5750	0.0750	0.0750
4/6	0.2592	0.7407	0.0741	0.0741
5/6	0.1157	0.8842	0.0509	0.0509
Total	2.2082	2.7916	0.2917	0.2917
		$n=5$		
1/5	0.7631	0.2369	0.0369	0.0369
2/5	0.5354	0.4646	0.0646	0.0646
3/5	0.3262	0.6738	0.0738	0.0738
4/5	0.1446	0.8554	0.0554	0.0554
Total	1.7693	2.2307	0.2307	0.2307
		$n=4$		
1/4	0.7074	0.2926	0.0426	0.0426
2/4	0.4318	0.5682	0.0682	0.0682
3/4	0.1903	0.8097	0.0597	0.0597
Total	1.3295	1.6705	0.1705	0.1705
		$n=3$		
1/3	0.6173	0.3827	0.0494	0.0494
2/3	0.2716	0.7284	0.0617	0.0617
Total	0.8889	1.1111	0.1111	0.1111

In order to facilitate the computation of the reactions the preceding tables have been prepared for girders with equal panels. These will be found sufficient for many structures. For bridges not covered by these tables the formulas previously developed should be employed.

202. Maximum Stresses in Swing Bridges. *Assumed Conditions.* To determine the maximum stresses in swing bridge trusses, the stresses due to several conditions of loading must be determined. These conditions may be stated as follows:

a. Dead load with bridge open; i.e., with end reactions equal to zero.

b. Snow load, bridge open. The weight to be assumed for the snow depends upon the situation of the bridge. An allowance of 10 lbs. per sq. ft. is probably a reasonable one for the latitude of New York or Boston. A snow load should be used only for highway bridges and railroad bridges with solid floors.

c. Live load only; truss to be considered either continuous or partially continuous as its construction may warrant.

d. Dead load with bridge closed and with end reactions acting upward, each equal to twice the maximum live end uplift at the end where applied. This assumption for the reactions is to prevent end hammer by providing liberally for impact, effect of changes in temperature, and wear in end-lifting apparatus. The end-lifting apparatus should be so designed that neither end of the structure can be lifted a distance greater than the upward deflection at that end due to the application there of a concentrated load equal to the assumed value of the dead reaction.

e. Live load only; truss to be considered as supported at one end and at the centre point of support adjoining this end, thus acting as a simple span. This condition may occur if the ends are not raised after the bridge is closed, either by carelessness or because of breakdown in the end-lifting apparatus.[1]

The maximum stress for any given bar will be either that due to Cases *a* and *b* combined, or that due to any reasonable combination of live and dead loading. Such combinations may be as follows, it being understood that impact should be added in each case in accordance with appropriate specifications.

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[1] If one end only is raised its normal amount, no dead reaction will occur at either end, the bridge being simply tilted.

1st. Dead stress, Case *d*, and live stress, Case *c*.

2nd. Dead stress, Case *a*, and live stress, Case *c*. This condition may exist if the ends are merely supported at the abutments when the bridge is closed and not actually raised, and if the live load be so applied that no end uplift will occur. The designer should observe that a live load may be discontinuous and that hence a full panel load may be considered at either end with live panel loads at any or all other points of the structure, and that this end panel load may be sufficient to overcome uplift due to the other loads. The designer must use his judgment in determining whether the conditions described in this paragraph are likely to occur for the loading giving maximum stress in any given bar of the truss under consideration.

3rd. Dead stress, Case *a*, and live stress, Case *e*. This condition may occur if the ends are not raised and if an unbalanced uplift exists at one end.

203. Specifications for Impact and Reversal of Stress. Opinions as to the proper method of making allowances for impact and for reversal of stress due to motion of bridge vary. The following quotations are indicative of these variations:

From Proceedings of the American Society of Civil Engineers of March, 1913:

"In calculating the dead-load stresses in the moving structural parts, for the various positions of the open bridge, such stresses shall be increased 25 per cent. as allowance for impact. For stationary structural parts (as towers, and supporting girders in rolling bridges), to which moving structural parts are attached, or on which such parts roll, 15 per cent. of the static stress shall be added as impact.

"In structural steel parts, where a percentage of the dead load or static stress is added for impact, the unit stresses for stationary structures shall be used; the impact percentages are an allowance similar to that provided by an impact formula for stationary railroad bridges.

"In structural members subject to reversal on account of the motion of the span, the effect of reversal shall be neglected. The member shall be designed for the stress giving the larger section. For riveted connections, the number of rivets shall be increased 25 per cent. over that required for the static stress plus impact stress.

"The allowance for impact in trunnions, cables, cable attachments, machinery parts, and structural parts supporting the machinery, is taken care of by lowered unit stresses."

From Specifications for Bridges Carrying Electric Railways Adopted by the Massachusetts Public Service Commission, March, 1915, Sections 18 and 20:

"On swing bridges and other movable structures of the dead load of the moving parts during motion shall be increased 10 per cent., and the increased stresses shall be considered the actual dead-load stresses.

"If a piece is exposed to both tension and compression, it must be proportioned to resist the maximum of each kind; but the unit stresses shall be less than those used for stress of one kind and shall be determined by multiplying the allowable unit stresses in Art. 17 (unit stresses for ordinary conditions) by the quantity $1 - \frac{\text{minimum stress}}{\text{twice the maximum stress}}$.

204. Computation of Maximum Stresses in Swing Bridges by Approximate Method. *Illustration.*

Problem. Compute the maximum stress in all bars of the truss shown in Fig. 320 for the following loads.[1]

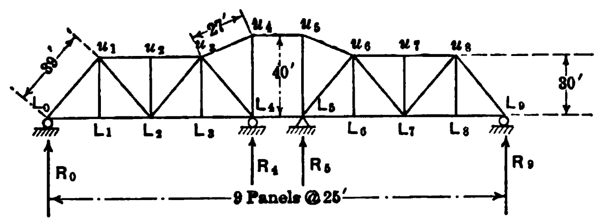


FIG. 320.

Dead—600 lbs. per ft. all on bottom chord, giving panel loads equal to 15,000 lbs.

Snow—No snow load. Bridge has open floor.

Live —2000 lbs. per ft. on bottom chord, giving panel loads equal to 50,000 lbs., and locomotive excess of 40,000 lbs.

Solution.

Case a. The index stresses in 1000-lb. units for all bars to left of U_3L_4 are shown in Fig. 321. The actual stresses in these bars are given in the table which follows. In determining these stresses, the end panel loads are assumed to equal the panel loads at intermediate points in order to provide for the weight of the end-lifting apparatus.

[1] Note that a span as short as this would ordinarily be constructed of type *a*, Art. 19. Type *c* is used in this problem as it is slightly more complicated to compute and hence furnishes a better illustration of methods of computation.

The stresses or their components in the other bars are given by the following computations:

Bars U_3U_4 and U_4U_5. Use method of moments with L_4 as origin.

$$\text{Horizontal component} = \frac{15(1+2+3+4)25}{40} = +93.75.$$

Bar L_4L_5. Stress $= -93.75$.

Bar U_4L_4. Stress $=$ vertical component in U_3U_4.

$$= -93.75 \times \frac{10}{25} = -37.50.$$

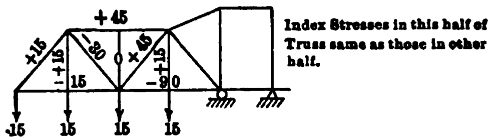


FIG. 321.

Bar U_3L_4. Use method of moments with vertical section between L_3 and L_4 and origin at intersection of U_3U_4 and bottom chord which occurs at L_0.

$$\text{Vertical component} = -15\ (1+2+3)\frac{25}{100} = -22.5.$$

DEAD STRESSES, CASE A, IN UNITS OF 1000 POUNDS

Bar.	Index Stress or Computed Component.	Ratio.	Actual Stresses.
L_0L_1, L_1L_2	−15.0	25/30	− 12.5
L_2L_3, L_3L_4	−90.0	25/30	− 75.0
U_4U_5	+93.8	....	+ 93.8
U_3U_4	+93.8	27/25	+101.3
L_4L_5	−93.8	....	− 93.8
U_4L_4	−37.5	....	− 37.5
U_1U_2, U_2U_3	+45.0	25/30	+ 37.5
L_0U_1	+15.0	39/30	+ 19.5
U_1L_2	−30.0	39/30	− 39.0
L_2U_3	+45.0	39/30	+ 58.5
U_3L_4	−22.5	39/30	− 29.3
U_1L_1, U_3L_3	+15.0	....	+ 15.0
U_2L_2	− 0.0	....	0.0

The actual stresses in all bars for this case and the reactions at all points are given in the preceding table:

Reactions at L_4 and L_5 are equal to one-half total dead load $=75$; other reactions are zero.

Case b. This case need not be considered as snow load is negligible.

Case c. The truss is partially continuous, hence the values in Table 3, Art. 201, will be used in computing the reactions. The position of the loads for any given bar may be readily determined by use of the reactions given in the table, and such few computations as are necessary will not generally be given.

The maximum values of uplifts and reactions will first be computed.

Maximum End Uplifts. Inspection of table shows that maximum uplift at L_0 occurs with full load from L_5 to L_9 and with E at L_7. Its value $=0.171\times50+0.068\times40=11.27$.

This equals the maximum uplift at L_9 which will occur with full load from L_0 to L_4 and with E at L_2. This is so small that a fraction of the full live panel load at the end would balance it, hence Case *c* may occur even if ends are not raised; that is, Case *a* and Case *c* may be combined.

No uplift will occur at points L_4 and L_5 since a load upon any portion of the structure will cause upward reactions at these supports.

Maximum Reactions. The maximum gross upward reaction at L_0 will occur with loads from L_0 to L_4 inclusive and with E at L_0. Its value $=$ $1.330\times50+50+40=156.5$. This equals the maximum gross reaction at L_9.

The maximum gross upward reaction at L_4 occurs with full load from L_0 to L_9 inclusive with E at L_4. Its value $=1.841\times50+50+40=$ 182.0. This equals the maximum upward gross reaction at L_5.

The loading and necessary computation for maximum stresses in all the bars of this truss for all possible combinations are given in the following table:

MAXIMUM LIVE STRESS, CASE C, IN UNITS OF 1000 LBS.

Bar.	Position of Uniform Load.	Position of E.	All Necessary Stress Computations. R_L = left reaction (+) when upward. V.C. = vertical component (+) when tension. H.C. = horizontal component (+) when tension. S = maximum stress.		
L_0L_1	L_0 to L_4 incl.	L_1	$R_L = +1.330\times50+0.707\times40$	= +	94.8
			$S = 94.8\times\frac{25}{30}$	= +	79.0
L_1L_2	L_5 to L_9 incl.	L_7	$R_L = -(0.171\times50+0.068\times40)$	= −	11.3
			$S = -11.3\times\frac{25}{30}$	= −	9.4
L_2L_3 and L_3L_4	L_0 to L_4 incl.	L_3	$R_L = 1.330\times50+0.190\times40$	= +	74.1
			S $= +74.1\times\frac{75}{30}-\frac{50\times75}{30}$	= +	60.3
	L_5 to L_9 incl.	L_7	$R_L = -11.3$		
			$S = -11.3\times\frac{75}{30}$	= −	28.2
U_3U_4	L_0 to L_9 incl.	L_7	$R_L = (1.330-0.170)50-0.068\times40$	= +	55.3
			H.C. $= +\frac{50\times6\times25-100\times55.28}{40}$	= +	49.3
			$S = 49.3\times\frac{27}{25}$	= +	53.2
U_4U_5	L_0 to L_9 incl.	L_7	+49.3		
L_4L_5	L_0 to L_9 incl.	L_7	−49.3		
U_4L_4	L_0 to L_9 incl.	L_7	$-49.3\times\frac{10}{25}$	= −	19.7
U_1U_2 and U_2U_3	L_0 to L_4 incl.	L_2	$R_L = 1.330\times50+0.432\times40$	= +	83.8
			$S = \frac{83.78\times50-50\times25}{30}$	= −	98.0
	L_5 to L_9 incl.	L_7	$R_L = -11.3$		
			$S = +11.3\times\frac{50}{30}$	= +	18.8
L_0U_1	L_0 to L_4 incl.	L_1	$R_L = +1.330\times50+0.707\times40$	= +	94.8
			V.C. $= -94.8$		
			$S = 94.8\times\frac{39}{30}$	= −	123.2

MAXIMUM LIVE STRESSES, CASE C, IN UNITS OF 1000 LBS.—*Continued*

Bar.	Position of Uniform Load.	Position of R.	All Necessary Stress Computations. R_L = left reaction (+) when upward. V.C. = vertical component (+) when tension. H.C. = horizontal component (+) when tension. S = maximum stress.		
L_0U_1	L_5 to L_9 incl.	L_7	$R_L = -11.3$		
			V.C. $= +11.3$		
			$S = 11.3 \times \frac{39}{30}$	= +	14.6
U_1L_2	L_2 to L_4 incl.	L_2	V.C. $= 50(0.432+0.190)+40\times0.432$	= +	48.4
			$S = 48.4\times\frac{39}{30}$	= +	62.9
	L_0 to L_1 incl. and L_5 to L_9 incl.	L_1	V.C. $= 90(1.000-0.707)+50\times0.170$	= −	34.9
			$S = -34.9\times\frac{39}{30}$	= −	45.3
L_2U_3	L_3 to L_4 incl.	L_3	V.C. $= 90\times0.190$	= −	17.1
			$S = -17.1\times\frac{39}{30}$	= −	22.2
	L_0 to L_2 incl. and L_5 to L_9 incl.	L_2	V.C. $= 50(1.000+0.170-0.707)+90(1.000-0.432)$	= +	74.3
			$S = 74.3\times\frac{39}{30}$	= +	96.6
U_3L_4	L_0 to L_3 incl.	L_3	V.C. $= 50\left(\frac{75}{100}\right)+90\left(\frac{75}{100}\right)$	= −	105.0
			$S = -105\times\frac{39}{30}$	= −	136.5
U_1L_1 and U_3L_3	L_0 to L_2 incl.	L_1		+	90
	L_2 to L_4 incl.	L_3		+	90
U_2L_2					0

Case d. The maximum live end uplift has been found to be 11.3, hence use a dead reaction of 2 × 11.3 or 22.6. The dead stresses may be most readily determined by subtracting the stresses due to this end reaction from the dead stresses already found. The necessary computations and final results are given in the following table:

DEAD STRESSES, CASE *D*, IN UNITS OF 1000 LBS.

Bar.	Components due to Reaction.	Stresses due to Reaction.	Stresses Case *a*.	Actual Stresses Case *d*.
L_0L_1, L_1L_2	H.C. = $22.5\times\frac{25}{30}$ = +18.75	+18.8	− 12.5	+ 6.3
L_2L_3, L_3L_4	H.C. = +56.25	+56.3	− 75.0	−18.7
U_3U_4	H.C. = $-22.5\times\frac{100}{40}$ = −56.25	−60.8	+101.3	+40.5
U_4U_5	H.C. = −56.25	−56.3	+ 93.8	+37.5
L_4L_5	H.C. = +56.25	+56.3	− 93.8	−37.5
U_4L_4	V.C. = $+56.25\times\frac{10}{25}$ = +22.5	+22.5	− 37.5	−15.0
U_1U_2, U_2U_3	H.C. = −37.5	−37.5	+ 37.5	0.0
L_0U_1	V.C. = −22.5	−29.3	+ 19.5	− 9.8
U_1L_2	V.C. = +22.5	+29.3	− 39.0	− 9.7
L_2U_3	V.C. = −22.5	−29.3	+ 58.5	+29.2
U_3L_4	V.C. = 0.	0.0	− 29.3	−29.3
U_1L_1, U_3L_3	V.C. = 0.	0.0	+ 15.0	+15.0
U_2L_2	0.	0.0	0.0	0.0

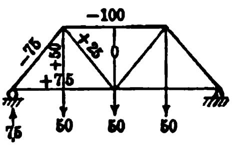


FIG. 322.

Case e. Index stresses for this case are shown in Fig. 322. Actual stresses are given in the following table:

MAXIMUM LIVE STRESSES, CASE E, IN UNITS OF 1000 LBS.

Bar.	Index Stress or Computed Component due to Uniform Live Load.	Ratio.	Actual Stress due to Uniform Load.	Stress due to *E*.	Maximum Stress.
L_0L_1, L_1L_2, L_2L_3, L_3L_4	+ 75	25/30	+62.5	$+\frac{3}{4}(40)\frac{25}{30}$ = +25.0	+ 87.5
U_1U_2, U_2U_3	−100	25/30	−83.3	$-\frac{1}{2}(40)\frac{50}{30}$ = −33.3	−116.6
L_0U_1, U_3L_4	− 75	39/30	−97.5	$-\frac{3}{4}(40)\frac{39}{30}$ = −39.0	−136.5
U_1L_2	$+\frac{3}{4}50 = +37.5$	39/30	+48.8	$+\frac{1}{2}(40)\frac{39}{30}$ = +26.0	+ 74.8
L_2U_3	$-\frac{1}{4}50 = -12.5$	39/30	−16.3	$-\frac{1}{4}(40)\frac{39}{30}$ = −13.0	− 29.3
U_1L_1			+50.0	+40	+ 90.0
U_3L_3			+50.0	+40	+ 90.0
All other bars = 0.					

TABLE GIVING MAXIMUM STRESSES COMBINED WITH IMPACT *

Bar.	Case a.		Case c.	Case d.	Case e.	Combination for Maximum.	Maximum Stress.
	Dead I.	Dead + Impact II.	Live + Impact III.	Dead IV.	Live + Impact V.		
L_0l_1	− 12.5	− 15.6	−9.4×1.75 = −16.5	+ 6.3	+87.5×1.75 = +153.1	I & III	− 29.0
L_1l_2			+79.0×1.75 = + 138.2			III & IV	+144.5
L_2L_3	− 75.0	− 93.8	+60.3×1.75 = +105.5	−18.7	+153.1	I & III	−124.4
L_3L_4			−28.2×1.75 = −49.4			III & IV	+ 86.8
U_3U_4	+101.3	+126.6	+42.6×1.57 = +66.9	+40.5	0	I & III	+168.2
U_4U_5	+ 93.8	+117.3	+39.4×1.57 = 61.9	+37.5	0	I & III	+155.7
L_4L_5	− 93.8	−117.3	−61.9	−37.5	0	I & III	−155.7
U_4L_4	− 37.5	− 46.9	−15.8×1.57 = −24.8	−15.0	0	I & III	− 62.3
U_1U_2	+ 37.5	+ 46.9	+18.8×1.75 = +32.9	0.0	−116.6×1.75 = −204.1	I & III	+ 70.4
U_2U_3			−98.0×1.75 = −171.5			III & IV	−171.5
L_0U_1	+ 19.5	+ 24.4	+14.6×1.75 = +25.5	− 9.8	−136.5×1.75 = −238.9	I & III	+ 45.0
			−123.2×1.75 = −215.6			III & IV	−225.4
U_1L_2	− 39.0	− 48.8	+63×1.86 = +117.2	− 9.7	+74.8×1.86 = +139.1	III & IV	+107.5
			−45.3×1.71 = −77.5		−29.3×1.92 = −56.3	I & III	−116.5
L_2U_3	+ 58.5	+ 73.1	−22.2×1.92 = −42.6	+29.2	−139.1	I & III	+219.8
			+96.6×1.67 = +161.3		+56.3	I & V	−80.6
U_3L_4	− 29.3	− 36.6	−136.5×1.80 = −245.7	−29.3	−238.9	I & III or III & IV	−275.0
U_3L_3	+ 15.0	+ 18.8	+90×1.86 = +167.4	+15.0	+167.4	III & IV	+182.4
U_2L_2	0.0	0.0	0.0	0.0	0.0		0.0

* Dead impact due to motion of bridge assumed at 25 per cent.

Live impact by formula (7), $I = S\dfrac{300}{L+300} = 0.75$ with one arm fully loaded.

$= 0.57$ with both arms fully loaded.

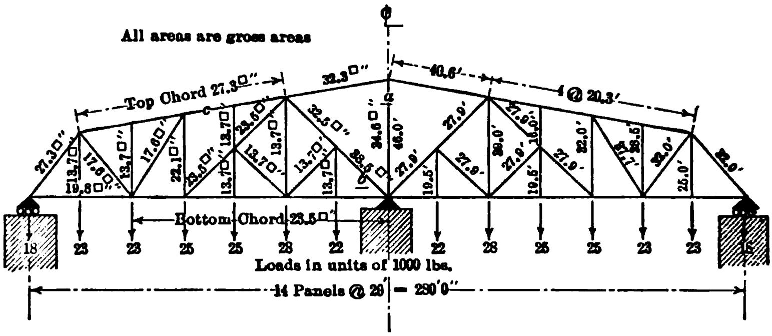
Reactions,

End	156.5 × 1.75	= 273.9
Intermediate	182.0 × 1.57	= 285.7

Uplift,

End	11.3 × 1.75	= 19.8

Problem 68. *a.* Compute deflection downward due to dead panel loads shown on figure of each end of this truss when bridge is open.



PROB. 68 AND 69.

b. Compute upward deflection at each end due to simultaneous application of upward forces of 33,000 lbs. at each end.

Problem 69. Compute maximum live stresses in bars *a*, *b*, and *c*, assuming structure to act as a continuous girder. Use loading of 1500 lbs. per lineal foot.

CHAPTER XVIII

MASONRY DAMS

205. Definitions. A dam is a structure designed to resist water pressure either by its weight alone, in which case it is called a gravity dam, or by its weight and resistance to bending combined, as in the case of a reinforced concrete dam. In either type, the resultant of the water pressure and the weight of the dam must pass through the base of the dam at a safe distance from its edge as explained later. An arched dam is one that is curved in plan and in which arch action as well as gravity may be counted upon to resist the water pressure.

Dams may be divided into two general classes:

a. Reservoir dams with the top of the dam at a level always higher than the water surface at the back of the dam.

b. Overflow dams with the top of the dam at a level lower than the maximum height of the water at the back of the dam.

The former type only will be considered here, but the same general principles are applicable to both classes, the overflow dam differing only in having a head of water at its crest and possibly a vacuum between the sheet of falling water and the downstream surface of the dam.

206. Assumptions for Gravity Dams. The design of gravity dams is ordinarily based upon the following limitations and assumptions:

1st. The portions of the structure above and below any assumed horizontal plane[1] act as monoliths.

2d. Tension may not exist upon any horizontal plane.

3d. Plane sections through the dam remain plane during distortion of structure.

4th. Stress varies as strain.

207. Distribution of Stress over Joints of Masonry Dams. The intensity of stress at the extreme fibre of a plane horizontal

[1] Such a horizontal plane is commonly called a joint regardless of whether it coincides or not with an actual joint in the masonry and will be so designated hereafter.

section of a block of homogeneous material capable of resisting both tension and compression is given by the following well-known equation:

$$s = \frac{P}{A} \pm \frac{Mc}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (68)$$

in which

s = maximum unit stress in lbs. per sq. ft.

P = resultant in lbs. of the vertical forces above the section.

A = area of section in square feet.

M = moment in ft. lbs. about neutral axis[1] of the section of the external forces applied to the portion of the structure above the section, combined with the moment about the same axis due to the weight of that portion of the structure lying above the given section.

c = distance in ft. from neutral axis to the extreme fibre of the section.

I = moment of inertia of section in ft.^{4} about neutral axis.

If Equation 68 be applied to the case of a rectangular joint one ft. in width and d ft. in length it becomes:

$$s = \frac{P}{d} \pm \frac{6M}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

The above equation may be used to determine the distribution of stress over the joint of a masonry structure which is incapable of carrying tension provided the value of s is never negative. The limitations which this latter condition imposes are explained later.

208. Application of Equations to Dams. Fig. 323 illustrates the forces acting upon a portion of a dam.

M_1 = resultant moment about central axis of base due to water pressure acting upon the right-hand surface of the entire dam above base of section shown combined with that upon base if upward pressure is assumed, and P = weight of entire dam above base of section shown. Evidently $M_1 - Pz = M$ of previous formulas.

Instead of using the moment M_1 due to the water, the resultant of the water pressure and the weight of the dam may be deter-

[1] The neutral axis as here used is the neutral axis of the cross-section with respect to flexure only; i.e., it is the principal axis of the section lying parallel to the longitudinal axis of the dam.

mined and the point where this resultant cuts the joint located. The moment of the resultant may then be computed and the maximum fibre stress determined as follows:

Let the vertical component of this resultant be V, and let the distance from its point of intersection with the joint to the neutral axis of the joint be x, then the general formula for maximum intensity of stress becomes,

$$s=\frac{V}{A}\pm\frac{Vxc}{I}.$$

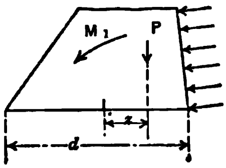


FIG. 323.

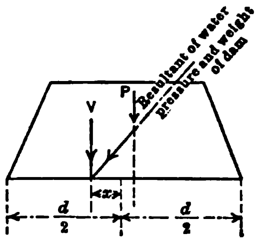


FIG. 324.

For a rectangular joint of width unity and length d, this may be written:

$$s=\frac{V}{d}\pm\frac{6Vx}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

This case is illustrated by Fig. 324.

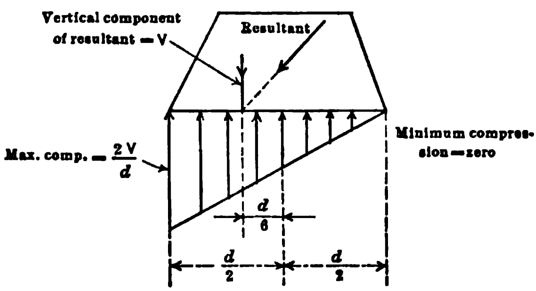


FIG. 325.

As the assumptions of Article 206 are not strictly correct for a material like masonry, formulas 68 to 70 incl. are somewhat approximate, but are in general use and are probably as accurate as the character of the data available in problems of dam design will warrant.

Examination of formula 70 shows that the stress at one edge of the joint will equal zero when $\frac{V}{d}-\frac{6Vx}{d^2}=0$, i.e., when $x=\frac{d}{6}$; and that when x is greater than $\frac{d}{6}$ there will be a tendency for tension to occur at one edge of the joint. The following important theorem may therefore be stated:

In order that tension may not exist at any point of a rectangular masonry joint, the resultant pressure on the joint must lie within its middle third.

Further consideration of this formula shows that when $x=\frac{d}{6}$ the maximum pressure on the joint $=\frac{V}{d}+\frac{6Vd}{6d^2}=\frac{2V}{d}=$ twice the average pressure. That is, for a joint where the resultant passes through the middle third point the maximum compression is twice the average and the minimum is zero. This is illustrated graphically by Fig. 325.

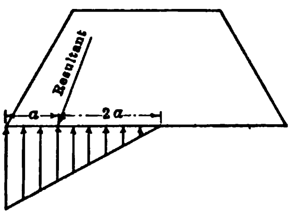


FIG. 326.

It is also evident that if the resultant passes outside of the middle third of the entire joint and if the material cannot resist tension, the compression will be distributed over only that portion of the joint which has its middle third point at the point of application of the resultant, the remainder of the joint offering no resistance to bending. Such a distribution of stress is shown in Fig. 326.

209. Outer Forces. The outer forces to be considered require the most careful study, especially for high dams. In general it would seem as if two cases only need be treated, reservoir full and reservoir empty, but the question of what, if any, upward[1] water pressure should be assumed under the dam and at the

[1] Much difference of opinion exists amongst engineers as to what allowance should be made for upward pressure, and for discussions on this point the reader is referred to references at close of chapter. It may be safely stated, however, that the absolute exclusion of upward water pressure can only be accomplished by the use of non-porous material laid in the dry in waterproof mortar, on a non-porous base, and that the amount of water pressure which will actually occur varies directly with the porosity of foundation, cementing material and masonry.

various joints, and what ice pressure should be considered must be thoroughly investigated.

To illustrate the loading used for one of the most important dams in the United States the conditions assumed in the design of the Wachusett Dam of the Metropolitan Water Works of Massachusetts as given on the official drawing of the accepted section are given below:

"1. Reservoir empty, i.e., water drained to elevation 300."

"2. Reservoir filled to elevation 400. No ice thrust and no upward water pressure included."

"3. Reservoir filled to elevation 395. Ice pressure 47,000 lbs. per linear foot of dam, at elevation 395. Upward water pressure corresponding to reservoir head at heel of joint and to backwater head at toe, varying uniformly from heel to toe, uniformly distributed, and considered to be exerted on two-thirds area of joint. Water is assumed to press against dam, where earth is filled against it, in same manner as if earth were not there. Only vertical pressure of earth over the masonry has been included, due allowance being made for diminished weight of particles of earth when submerged."

210. Economical Cross-section. The economical profile of a masonry dam may be determined in the following manner by

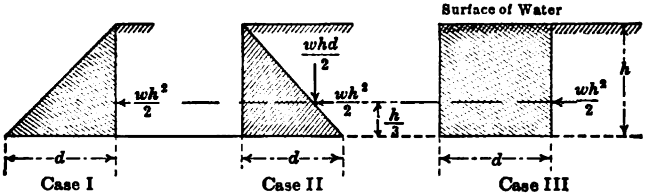


FIG. 327.

use of the assumptions previously stated, provided the effect of upward pressure be neglected. Let the various profiles shown in Fig. 327 be considered. Let the weight of the masonry per cu. ft. $= \gamma$ and the weight of the water per cu. ft. $= w$. The width of base consistent with no tension may then be computed in terms of h and d. The horizontal component of the water pressure is the same in all cases and is shown on the figure. For Case II a vertical component also exists equal to $\frac{whd}{2}$.

The resultant of the dead load acts at the following points:

Case I. Inner middle third point.

Case II. Outer middle third point.

Case III. Center of section.

In order to obtain the maximum economy consistent with no tension on the base, the resultant must evidently pass through the *outer middle third* point (considering the *outer* side to be that farthest away from the water). The moment of the resultant under these conditions about the middle third point equals zero, equals the combined moment of the weight of the dam and the water pressure, hence the width of base may be obtained by putting the expression for the moment about the outer middle third point equal to zero.

The resulting equations are as follows:

$$\text{Case I.} \left(\frac{\gamma hd}{2}\right)\left(\frac{d}{3}\right)-\left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right)=0$$

$$\text{Case II.} \left(\frac{whd}{2}\right)\left(\frac{d}{3}\right)-\left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right)=0$$

$$\text{Case III.} (\gamma hd)\left(\frac{d}{6}\right)-\left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right)=0$$

Solving these equations gives the following results:

$$\text{Case I.}\ d=h\sqrt{\frac{w}{\gamma}}$$

$$\text{Case II.}\ d=h$$

$$\text{Case III.}\ d=h\sqrt{\frac{w}{\gamma}}$$

Since w is always less than γ it is evident that Case I gives a more economical section for full reservoir than either of the other cases. It should be noticed that this case also gives the limiting condition when the reservoir is empty, the resultant pressure then passing through the inner middle third point.

$\frac{\gamma}{w}$ = specific gravity of the masonry, hence if this be denoted by β we may write for Case I,

$$d=\frac{h}{\sqrt{\beta}} \qquad \text{or} \qquad h=d\sqrt{\beta}$$

The section shown by Case I cannot be adopted in practice

since an appreciable top width is necessary to resist the action of waves, ice and floating material of all sorts, and to serve as a foot-path or driveway. Moreover, the top of the dam for the same reasons should extend somewhat above the normal water level. At the bottom of the dam hydraulic pressure may also occur on the downstream side and on the base. It is also necessary to consider the allowable pressure at the base of the dam which for high dams is often the controlling factor in determining the width of base. The resistance of the dam to slipping on any joint must also be considered, and the section so proportioned that the resultant pressure at any joint will not make an angle with the vertical greater than the angle of repose at that joint.

Regard for the above-mentioned considerations coupled with the necessity of giving a pleasing section ordinarily results in

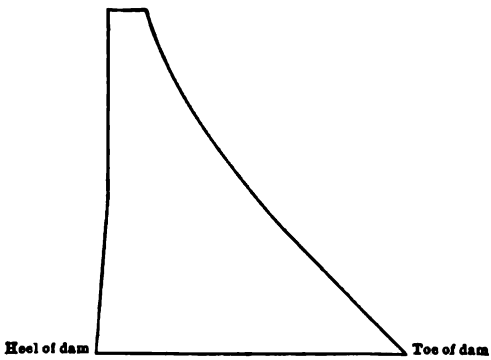


FIG. 328.

selecting for high reservoir dams a profile somewhat like that indicated in Fig. 328.

211. Determination of Profile of a Low Dam. If the dam under consideration is comparatively low, with a narrow top width, a trapezoidal profile may prove most economical. The width of base for such a dam may be readily determined analytically by the following simple method.

Let the width of base be assumed as equal to x, so that the weight of the dam itself as well as the upward water pressure can be expressed in terms of x. The water pressure on side bc (Fig. 329) can be expressed in terms of the known height h. Since the limiting case for stability will occur when the resultant

of all the forces acting on the dam passes through the outer middle third point e, the moment of these forces about e may be placed equal to zero and the resulting equation solved for x. The process is illustrated by the following example.

Problem. *a.* Determine analytically for the dam shown in Fig. 329 the width of base consistent with no tension. Assume masonry to weigh 150 lbs. per cu. ft. and upward pressure on base equal to two-thirds hydrostatic head at heel or upstream face reducing uniformly to zero at toe or downstream face.

b. Determine maximum intensity of pressure at base of dam.

c. Determine angle between resultant and normal to base.

Solution. Consider a slice of the wall one ft. in length perpendicu-

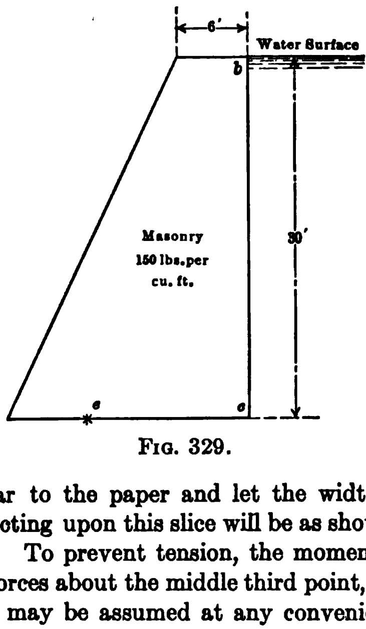


FIG. 329.

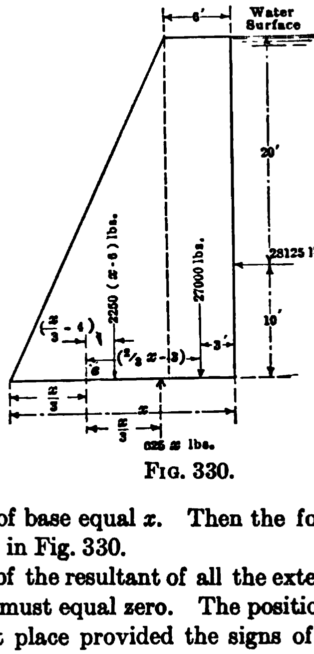


FIG. 330.

lar to the paper and let the width of base equal x. Then the forces acting upon this slice will be as shown in Fig. 330.

To prevent tension, the moment of the resultant of all the external forces about the middle third point, e, must equal zero. The position of e may be assumed at any convenient place provided the signs of the moments of the various forces about it are properly used. The equation resulting from applying $\Sigma M = 0$ about e is as follows:

$$27{,}000\left(\frac{2x}{3}-3\right)-28{,}125\times10-\frac{625x^2}{3}+2250\,(x-6)\left(\frac{x}{3}-4\right)=0$$

Solution of this equation gives $x = 20.1$ ft. which is the width of base required to prevent tension.

b. The maximum intensity of pressure may be obtained by applying

equation (69), M being the moment of all the applied forces about an axis passing through the centre of gravity of the base.

The maximum intensity of pressure at the base of the dam equals twice the average pressure since the resultant passes through the middle third point, hence its value $=\frac{2(27,000+2250\times14.1-62.5\times20.1)}{20.1}=4600$ lbs. per sq. ft.[1]

c. The maximum angle between resultant and normal to base occurs when upward pressure exists. Its tangent equals the total horizontal force divided by the total vertical force $=\frac{28,125}{46,160}=0.61$.

212. Determination of Profile of a High Dam. The determination of the profile of a high dam may be divided into several clearly defined cases which will be enumerated in order proceeding downward from the top of the dam.

Case 1. Sides vertical. Thickness equals that at top of dam. Limiting conditions—resultant, reservoir full, must not pass outside the outer middle third point of base of section.

Case 2. Inside face vertical, and outside face inclined. Limiting conditions—resultant, reservoir full, must not pass outside the outer middle third point; reservoir empty, must not pass inside the inner middle third point.

Case 3. Same as Case 2 but both faces inclined.

Case 4. Same as Case 3. Limiting conditions—pressure at toe of dam must not exceed the allowable unit stress;[2] resultant pressure, reservoir empty, must not pass inside the inner middle third point.

Case 5. Same as Case 3. Limiting conditions—pressure at toe, reservoir full, and at heel, reservoir empty, must not exceed allowable unit stresses.

In addition to the limitations imposed by above conditions it is also necessary to consider *stability against sliding of one portion*

---

[1] Note that this same value would be obtained if upward pressure were to be neglected and the intensity computed by the ordinary beam formula; since the intensity of the upward pressure has been assumed as zero at the toe of the dam.

[2] Note that allowable vertical pressure at toe of dam is usually taken as somewhat less than that elsewhere owing to the possibility that the maximum intensity of pressure occurs on an oblique instead of a horizontal joint. For an important dam the actual maximum intensity of pressure at the point should be determined. See article by Cain in Trans. Am. Soc. C. E., Vol. LXIV, pages 208 et seq.

of the dam on another. This condition will be satisfied if the angle between the resultant and normal at any section does not exceed the angle of repose of the material, a reasonable factor of safety being allowed. As dam failures have occurred through lateral sliding, this limitation should be carefully observed.

The determination of an exact profile to conform to all these conditions would prove a difficult problem, and is not attempted in practice. Instead, the assumed section of the dam may be divided into horizontal slices of trapezoidal section and of reasonable thickness, and the stability of each slice considered independently. The sides of the various trapezoids may then be connected by smooth curves to give the dam a graceful appearance.

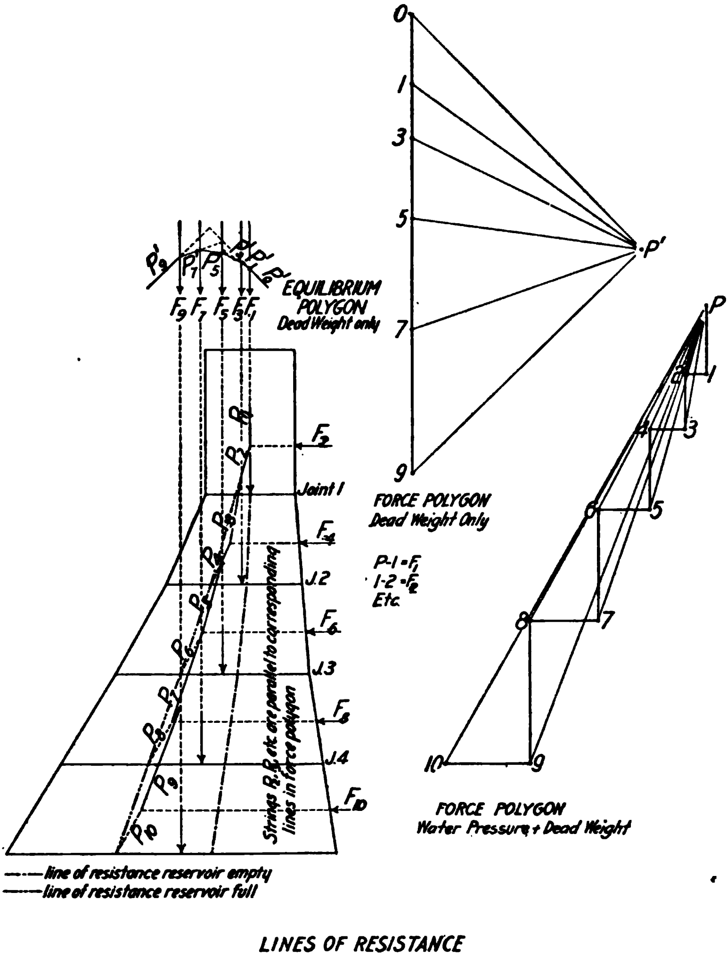
The depth of the slices into which the dam should be divided depends upon its height and must be settled by the judgment of the designer. A thickness of 10 ft. for a high dam may, however, be considered reasonable.

213. Graphical Method of Solution. While all the various cases of Art. 212 may be treated mathematically, some of the formulas are long and complicated and for simplicity either a purely graphical method or a combination of graphical and analytical methods may be used. The graphical method involves the construction of an equilibrium polygon for each condition of loading. The intersections of the appropriate strings with the horizontal planes at top and bottom of each slice into which the dam is assumed to be divided give the points of application of the resultant forces acting on these slices. A line connecting these points of intersection is called the line of resistance and should not pass outside the boundaries of the middle third of the cross-section. The resultant upon any joint should not make an angle with the normal to the joint greater than the angle of repose.

Fig. 331 shows the application of this process to a simple case. To construct the line of resistance, reservoir empty, requires the determination of the point of application of the resultant of the total weight above each joint. For clearness this is determined by a figure located above the dam profile. The various resultant weights F_1, F_3, etc., are projected up, the equilibrium polygon drawn, and the intersection of the strings P'_1, P_3^1, etc., with P'_0, projected downward to the joint above which the forces F_1, F_3, etc., act.

The line of resistance reservoir full is constructed by drawing

string P_2 through the intersection of F_1 and F_2 till it meets F_3, from this intersection P_3 is drawn till it meets F_4, etc. A line connecting the points of intersection of the equilibrium polygon strings with the various horizontal joints is the line of resistance.



LINES OF RESISTANCE
MASONRY DAM.

FIG. 331.

For each joint the point of intersection is that made by the string numbered to correspond to the horizontal force acting on the section of the dam immediately above the joint.

If upward pressure be assumed as occurring at the joints, it should be represented by an upward force at each joint which should be taken into account in constructing the equilibrium polygon.

The magnitude of the resultant upon any joint may be determined from the force polygon by scale; e.g., the resultant pressure on joint 3, reservoir full $= P_6$ of the force polygon.

The maximum intensity of pressure at any joint may be found when the position and magnitude of the resultant is known by the methods previously given for the distribution of stress over masonry joints.

It is evident that the graphical method is purely a method of trial. The fact that it is possible to start at the top and work downward, fixing the size of each trapezoidal slice in succession makes it much less tedious than if it were necessary to try an entire new profile each time.

In the application of the graphical method it is often desirable to determine graphically the centre of gravity of a trapezoid. This may be found by the following method:

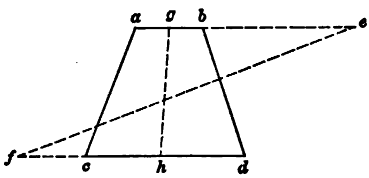


FIG. 332.

Prolong the parallel sides ab and cd Fig. 332 and lay off $be = cd$, and $cf = ab$. Connect g and h, the centres of sides ab and cd respectively, and e and f. The intersection of ef and gh will be at the centre of gravity of the trapezoid.

214. Graphical and Analytical Methods Combined. In order to reduce the number of trials necessary in the application of the graphical method, it is sometimes desirable to apply analytical methods to Cases 1 and 2, Art. 212, and these methods will now be given.

Case 1. The problem here is to determine the depth at which the width must begin to be increased to secure stability against overturning. The formula already deduced for the depth of a rectangular dam, $h = d\sqrt{\beta}$, may be used.

That the allowable compression will not be exceeded in this portion of the dam may be easily shown. Let the weight of the masonry be assumed as 156.25 lb., per cu. ft., β will then have a

value of $2\frac{1}{2}$ and h will equal 1.58 d. As this value of β is seldom exceeded and as d is seldom greater than 20 ft., h will rarely exceed 31.6 ft., hence the maximum compression on the masonry of this section will not generally exceed $2 \times 156.25 \times 31.6 =$ approximately 10,000 lbs. per sq. ft., a low value for masonry.

The above figures are determined for water pressure on side of dam only, the water being assumed to reach to the top of the dam. If ice pressure be considered, this result may be modified somewhat but thick ice would probably not occur with the maximum height of water. An upward water pressure occurring at any section would have the effect of reducing the weight of the masonry, i.e., of reducing β, and the depth of this portion of the dam should be reduced accordingly.

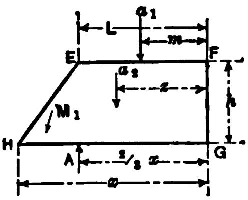


FIG. 333.

Case 2. The problem here is to determine the limit of depth for a vertical inner surface, and the necessary thickness of the dam at this limiting depth.

The following method referring to Fig. 333 may be used:

Let a_1 = area in sq. ft. of vertical slice of dam above upper joint (lower joint determined under Case 1).

m = distance from edge of this joint to a vertical line passing through center of gravity of a_1.

L = width of upper joint (previously determined).

a_2 = area of trapezoid $EFGH$ (length = unity).

z = horizontal distance from center of gravity of $EFGH$ to vertical side FG.

h = depth of $EFGH$.

M_1 = moment of horizontal forces applied above HG about any horizontal axis lying in joint HG and normal to trapezoid, divided by the weight of one cu. ft. of masonry.

x = width of joint HG.

A = resultant upward pressure on HG divided by the weight of one cu. ft. of masonry $= a_1 + a_2$.

Since the limiting condition occurs when the resultant upward pressure acts through the outer middle third point, and since

EFGH must be in equilibrium under the action of the external forces we may apply $\Sigma M = 0$ about the point G and solve for x. The following equation results:

$$a_1m+a_2z+M_1=\tfrac{2}{3}Ax$$

in which a_1, m, and M_1 are known, $a_2=h\left(\frac{L+x}{2}\right)$, and a_2z can be expressed in terms of h, L, and x.

To determine a_2z make use of the principle that the center of gravity of a triangle is coincident with the point of application of the resultant of a set of parallel forces applied in any direction at the *apices* of the triangle and each equal to $\frac{1}{3}$ its area. To apply this principle, divide the trapezoid into two triangles *EFH* and *EGH*, Fig. 330. The apex loads for triangle *EFH* are each $\frac{hL}{6}$; and for triangle *EGH*, $\frac{hx}{6}$.

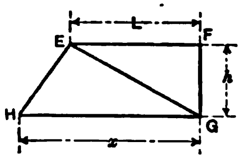


FIG. 334.

The moment of a_2 about G equals the moment of all of above apex loads about G equals $\left(\frac{hL}{6}+\frac{hx}{6}\right)L+\frac{hx}{6}\cdot x=\frac{h}{6}(L^2+Lx+x^2)=a_2z$. By substituting for a_2z the above value, and for A its value, viz., $a_1+a_2=a_1+\left(\frac{L+x}{2}\right)h$, the equation $a_1m+a_2z+M_1=\frac{2}{3}Ax$ becomes,

$$a_1m+\frac{h}{6}(L^2+Lx+x^2)+M_1=\frac{2x}{3}\left[a_1+\left(\frac{L+x}{2}\right)h\right]$$

which by reduction gives the following expression for the limiting value

$$x^2+x\left(L+\frac{4a_1}{h}\right)=\frac{6}{h}(M_1+a_1m)+L^2.$$

This equation gives the ratio between x and h for the limiting conditions, provided M_1 is also expressed in terms of h as may easily be done. The equation is somewhat complicated and most designers would prefer to use the graphical method throughout.

The above method is applicable, provided the compressive strength and resistance to horizontal sliding are not exceeded, until a point is reached where the line of pressure, reservoir empty, passes outside the middle third, from which point on a purely graphical method may be applied.

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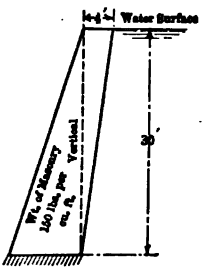
PROBLEMS

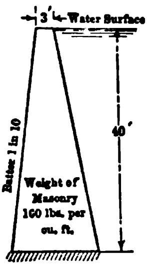
70. Determine width of base of this dam with water standing at level shown, assuming that no tension may exist and that no upward pressure occurs at base. Assume masonry to weigh 150 lbs. per cu. ft.

71. *a.* Determine width of base of this dam assuming that no tension may exist, and that upward water pressure acts on its base corresponding to full hydrostatic head at heel and zero at toe.

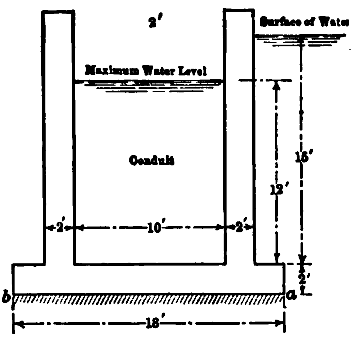
b. Compute maximum intensity of pressure at toe, and tangent of angle between normal and resultant at base.



PROB. 70.



PROB. 71.



PROB. 73.—Walls and base of reinforced concrete, wt. = 150 lbs.per cu. ft.

72. *a.* Draw a diagram showing the intensity of pressure across the base of the dam shown in Fig. 329 assuming its width to be 24 ft.

b. Draw a similar diagram assuming the width of base to be 18 ft.

73. Draw a diagram showing intensity and distribution of stress over the base of this conduit:

a. When conduit is full.

b. When conduit is empty.

Conduit is exposed to extreme water pressure on one side only as shown. Assume upward pressure on base corresponding to two-thirds of hydrostatic pressure at *a* and zero at *b*.

CHAPTER XIX

EARTH PRESSURE

215. Cohesion and Friction. In the theoretical treatment of earth pressure it is commonly assumed that the earth is a granular mass without cohesion acting like a pile of pebbles. As to the neglect of cohesion it may be said that while under some conditions a considerable amount of cohesion may exist in earth as is shown by the vertical slopes which frequently occur in freshly cut banks, its value is influenced greatly by the effect of moisture, and is often entirely destroyed by removing the earth from its original situation, hence in newly made embankments cohesion cannot be relied upon.

If cohesion does not exist, the surface slope of a mass of earth will make an angle with the horizontal, the tangent of which equals the coefficient of friction. The value of this coefficient varies with the character of the earth and with the amount of moisture which it contains; in railway construction it has been found necessary, for ordinary material, to use a slope of one and one-half horizontal to one vertical to prevent slipping. This is equivalent to using a coefficient of friction of two-thirds corresponding to an angle of repose of 33°–40′, a value which may probably be used with safety for most earth.

216. Active and Passive Pressure. In a fluid like water in which friction between the particles is zero the resultant pressure on any plane is normal to that plane, and can have but one value consistent with equilibrium. In a granular material, on the other hand, the resultant pressure on a given plane may make an angle with the normal less than or equal to the angle of repose and hence may have several values each of which is consistent with equilibrium in the material. This may be illustrated as follows:

Consider the equilibrium of a triangular prism *abc*, Fig. 331, contained in a mass of granular material like sand, the upper surface of the prism coinciding with the sloping upper surface of the sand. Assume that the pressure P on *ab* is parallel to the

surface but of unknown magnitude. For equilibrium the resultant pressure on the surface *bc* must make an angle with the normal, *ns*, less than the angle of repose φ of the material. Its two extreme positions are evidently *mn* and *no*, and it may lie anywhere between these two positions.

The two triangles of force shown by Figs. 336 and 337 represent the forces acting in each of the two extreme conditions of equilibrium.

In each case

W = weight of prism.
P = total force on side *ab*.
R = total force on side *bc*.

Fig. 336 shows the relative values of P and R when R corresponds in direction to *mn* and Fig. 337 shows the same values when R corresponds to *no*.

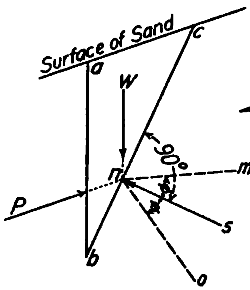


FIG. 335.

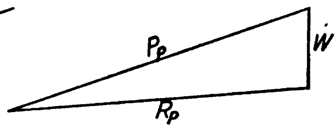


FIG. 336.

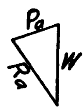


FIG. 337.

These diagrams show that the force P may vary considerably in magnitude without overcoming the equilibrium of the particle. P_a, the smaller value of P, is called the active pressure; it is the smallest force consistent with equilibrium of the particle, which can act in the assumed line of action of P; the application of a smaller force permits the resultant on the side *bc* to make an angle below the normal greater than φ and thus allows the prism to slide downward on the plane *bc*. P_a corresponds to the minimum force which a retaining wall holding back the earth to the right of *ab* must be designed to resist; it is called the active pressure since it equals the force which the earth actually exerts upon the wall.

P_p, the larger value of P, shown by Fig. 336, is the maximum value which may be exerted on ab without forcing the prism to slide up on the surface bc. It is called the passive pressure because it is a measure of the passive resistance of the earth to being forced upward. It corresponds to the force which could be counted upon to resist water pressure acting against ab, or to resist the overturning forces acting on a telegraph pole.

In order to determine the extreme values of P, in any given case, that is the minimum and maximum values consistent with equilibrium, it would be necessary to determine the value of the angle abc corresponding to these extreme cases as well as the points of application of the forces themselves. A common method in use for doing this is a method of trial which will now be explained.

217. Method of Trial. By this method force triangles are drawn for various planes passing through point b, Fig. 335. If the active pressure be desired the minimum value of P consistent with the resultant pressure on any plane bc making the angle φ below the normal is determined; for the passive pressure the corresponding value is obtained for the resultant making the angle φ above the normal. The pressure on any plane is assumed to vary uniformly downward from the top. The application of this method to a simple case is illustrated by Fig. 338 in which the planes $b1$, $b2$, etc., represent trial planes. The thrust on the back of the wall is in this case assumed to make an angle of 30° with the normal; i.e., the angle of repose between either earth and earth, or earth and wall is assumed to be 30°. The arcs shown are drawn in order to simplify the construction. These arcs are swung with equal radii from b and g_0 as centres. g_0s_1, g_0s_2, etc., show the assumed direction of the thrust on the planes $b1$, $b2$, etc., s_1s_2, s_2s_3, etc., are laid off on the arc swung with g_0 as a centre and are made equal to the intercepts between the lines $b1$, $b2$, etc., on the arc swung with b as a centre. This process is equivalent to laying off angles between the different normals equal to the angles between the planes corresponding to the different normals.

The distances g_0g_1, g_1g_2, etc., represent the weights of the various triangular prisms and may evidently be made equal to their bases 1 2, 2 3, etc., provided the scale be properly chosen.

The lines g_1f_1, g_2f_2, etc., are drawn parallel to the assumed direction of the pressure on the wall, and the points f_1, f_2, etc.,

are located at the intersection of these lines with the lines g_0s_1, etc. Through the f points a smooth curve is drawn and the maximum value of a line parallel to the gf lines, intercepted between the vertical through b and this curve is taken as equal to the maxi-

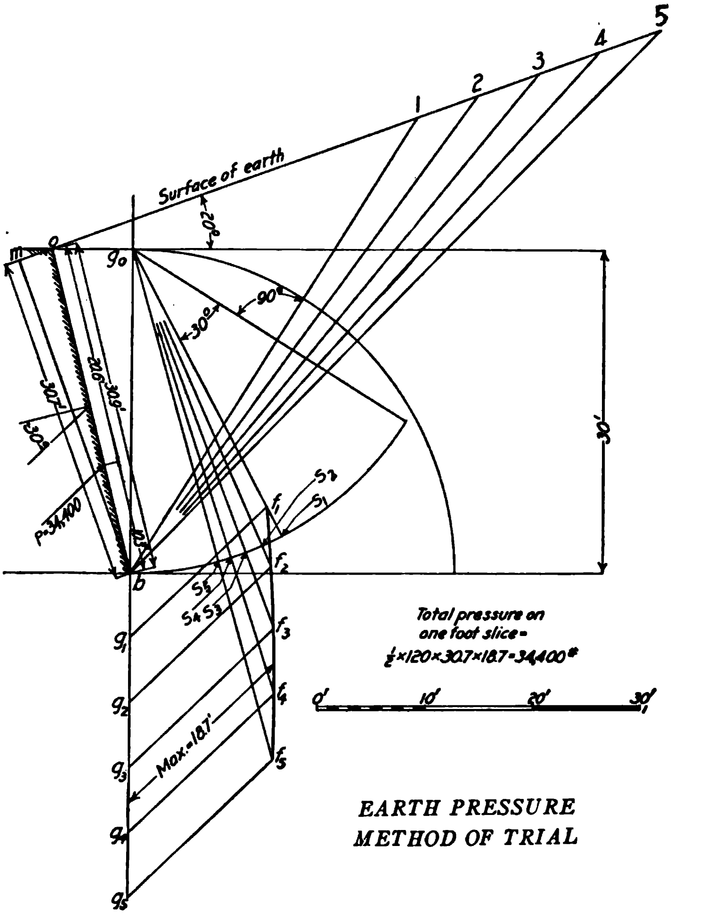


FIG. 338.

mum thrust. In the case shown this line as scaled on the scale of distance equals 18.7′. This must be multiplied by the scale of force which equals one-half the product of the common altitude, bm, of the various triangular prisms, and the weight per cu. ft. of the earth, assumed as 120 lbs. for this case.

218. Rankine's Method. Rankine's method is based primarily upon the assumption that the principles governing the distribution of stress in a homogeneous rigid body are applicable to a mass of earth. It is also based upon the following additional assumptions:

a. The mass of earth under consideration has a plane upper surface of unlimited extent.

b. The pressure on every plane is a thrust; i.e., tension or shear cannot occur on any plane.

c. The pressure on some one plane passing through the given point makes an angle with the normal to that plane just equal to the angle of repose of the material, and on no plane does it make an angle with the normal greater than this angle.

If these assumptions be made, it is possible to show that the stresses on two particular planes passing through a given point are conjugate stresses. (Stresses are conjugate when the stress on a given plane at a given point of a body is parallel to another plane the stress upon which is parallel to the first plane.) That this relation exists between the stress on a vertical plane and that on a plane parallel to the earth surface assuming above assumptions to be correct may be demonstrated as follows for points on the line of intersection of the two planes:

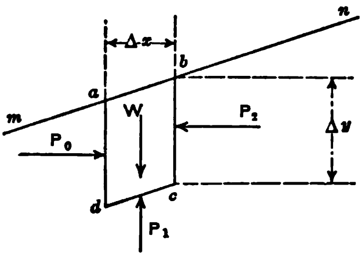


FIG. 339.

Let *mn* in Fig. 339 represent the earth surface and consider the relation between the stresses on two planes, one vertical and the other parallel to the earth surface, at point *d* at the intersection of the two planes.

Let *abcd* represent a particle of earth having a length of unity at right angles to the plane of the paper, and a weight of W acting as shown. Let the direct forces acting upon its sides be represented by P_0, P_1 and P_2 assumed to act as shown. By hypothesis there can be no other forces acting. For equilibrium all forces must meet at a point and this must evidently lie on the line of action of W, hence P_1 must be applied at centre of *dc*. Since the earth mass is considered of infinite extent, conditions on planes *ad* and *bc* must be identical, hence P_0 and P_2 must be

equal and parallel and act at equal distances from the surface, and as they must intersect on the line of action of W, they must be parallel to the earth's surface. As they are equal and parallel their horizontal components will be equal, hence P can have no horizontal component and must, therefore, be vertical.

It is furthermore clear that the conclusions just reached for an infinite mass are very nearly correct for a mass of earth with a plane upper surface of extent such that the conditions on two adjoining planes a slight distance apart are very nearly identical.

It follows that at any point in a mass of earth of reasonable extent and with plane upper surface, the pressure on a plane parallel to the surface is vertical, and the pressure on a vertical plane is parallel to the surface, hence these pressures are conjugate and the relation between the intensities of the pressures at any point upon two such planes may be expressed by the equations for conjugate forces. Let p and p' represent two such intensities. Their relation will then be expressed by the following equation.[1]

$$\frac{p}{p'}=\frac{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}}{\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}} \quad . \quad . \quad . \quad . \quad (71)$$

in which θ = the angle made by the surface with the horizontal and φ = angle of repose of the earth.

This formula is deduced for the case of p less than p'. If the hypothesis is made that p' is less than p, the first term of the equation should be inverted, the second remaining unchanged, hence we may write for the general case

$$\frac{p}{p'}=\frac{\cos\theta\mp\sqrt{\cos^2\theta-\cos^2\varphi}}{\cos\theta\pm\sqrt{\cos^2\theta-\cos^2\varphi}} \quad . \quad . \quad . \quad . \quad (72)$$

If p' represents the intensity of pressure on a plane parallel to the surface at a vertical distance y from it, its value will equal $wy \cos\theta$ lb. per square foot, in which w is the weight of a cubic foot of earth. Substituting this value in equation (72) gives the following range of values for p, the intensity of the pressure on a vertical plane at a vertical distance y below the surface.

$$p=wy\cos\theta\frac{\cos\theta\mp\sqrt{\cos^2\theta-\cos^2\varphi}}{\cos\theta\pm\sqrt{\cos^2\theta-\cos^2\varphi}} \quad . \quad . \quad . \quad (73)$$

[1] See Applied Mechanics, Lanza, 9th edition, page 889.

In this equation the negative sign in the numerator and the positive sign in the denominator should be used to get the active pressure; to get the passive pressure the signs should be reversed. The following inequalities may now be written to cover all possible cases of equilibrium.

$$p \geq wy \cos \theta \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} \quad . \quad . \quad . \quad (74)$$

$$p \leq wy \cos \theta \frac{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}} \quad . \quad . \quad . \quad (75)$$

These equations give the limiting values consistent with equilibrium of the intensities of pressure on a vertical surface; any value of p between these values will therefore be consistent with equilibrium. It is to be remembered that the direction of the pressure is parallel to the surface of the earth.

The total pressure on any vertical plane may be obtained by multiplying the values in equations (74) and (75) by $\frac{y}{2}$, giving the following formulas for active and passive pressures, respectively:

$$P_a = \frac{wy^2}{2} \cos \theta \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} \qquad (74a)$$

$$P_p = \frac{wy^2}{2} \cos \theta \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} \qquad (75a)$$

Since $wy \cos \theta$ varies directly with the distance y, p also varies with y having the value zero at the surface of the earth. The point of application of the resultant pressure on a vertical plane is, therefore, at a distance from the surface of the earth, equal to two-thirds the depth of the plane. When the earth surface is horizontal

$$p \gtrless wy \frac{1 \mp \sin \varphi}{1 \pm \sin \varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

when $\theta = \varphi$, $p = wy \cos \theta$ and the active and passive pressures are equal.

If φ = zero as in the case of water pressure

$$p = wy$$

By formulas 74*a* and 75*a* it is possible to determine the total pressure at any point on a *vertical* plane whether it be a vertical wall

surface or a vertical plane of earth. If it be desired to determine the pressure on an inclined wall it is necessary to compute the pressure on the vertical plane passing through the back corner of the wall and to combine this with the weight of the prism of earth between the wall and this plane. This is illustrated by the problem which follows.

Rankine's method is evidently inapplicable when a wall slopes backward as shown in Fig. 340, since for such a case the resultant of the weight of the prism *abc* and the pressure on *ac* as determined by Rankine's method may evidently make an angle with the normal to the wall greater than the angle of repose. For such a case the method of trial should be used.

It should be observed that Rankine's method makes no assumption as to the direction of the resultant pressure on an inclined surface. This may easily be determined, however, since the resultant of the pressure on a vertical plane is parallel to the earth surface and the resultant weight between the vertical plane and the surface of the wall is vertical. The following example illustrates the application of this method.

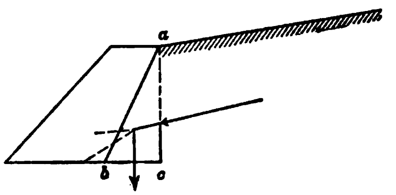


FIG. 340.

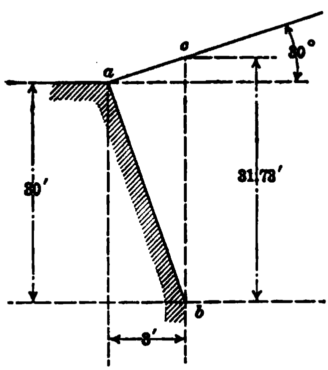


FIG. 341.

Problem. Determine the intensities of the active and passive pressures at point *b* on surface *ab* and the total active pressure per foot in length on that surface for the wall shown in Fig. 341.

Assume weight of earth = 100 lbs. cu. ft.
Assume weight of masonry = 150 lbs. cu. ft.
Assume φ = 35°.

Solution. $\cos\theta = 0.866 \qquad \cos^2\theta = 0.750$

$\cos\varphi = 0.819 \qquad \cos^2\varphi = 0.671$

$$\sqrt{\cos^2\theta - \cos^2\varphi} = 0.281$$

$$wy\cos\theta = 3173 \times 0.866 = 2748$$

Therefore, the two extreme conditions consistent with equilibrium are given by the following equations:

$$p = 2748 \frac{0.866 - 0.281}{0.866 + 0.281} = 2748 \times \frac{585}{1147} = 1402$$

and

$$p = 2748 \frac{0.866 + 0.281}{0.866 - 0.281} = 2748 \times \frac{1147}{585} = 5400$$

These two values of p are the minimum and maximum intensities of pressure at b consistent with equilibrium. The smaller is the active pressure.

The total pressure on bc for either condition of earth pressure may now be obtained by multiplying the average pressure on bc, i.e., one-half of either of the values just obtained, by the distance bc. The total active pressure on bc per foot of wall is thus found to be

$$31.73 \times \frac{1402}{2} = 22,250 \text{ lbs.}$$

This pressure on the vertical plane may be combined with the weight of the prism abc, and the result will be the pressure on the back of wall.

A graphical solution of Rankine's method is often advantageous to use particularly in determining the pressure upon various sections of a curved surface such as a tunnel wall. For a description of such a method see American Sewerage Practice, Vol. II.—Metcalf and Eddy.

219. Surcharged Wall. A mass of earth is surcharged when it carries an applied load such for example as a building or a railroad track and train as is illustrated by Fig. 342. In such a case the retaining wall must resist not only the horizontal

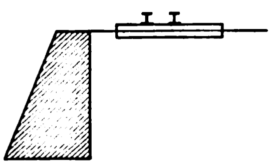


FIG. 342.

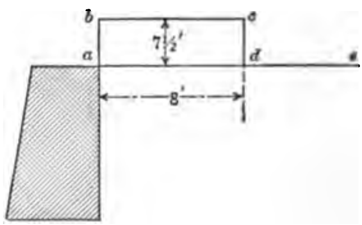


FIG. 343.

pressure due to the earth but also the additional pressure due to the superimposed load coming from the track and the train which it may carry.

Such cases are commonly treated by reducing the additional weight to an equivalent height of earth. For the case shown in

Fig. 342, the train load is assumed to have a maximum value of 4000 lbs. per lineal foot and the weight of track and ballast 2000 lbs. per linear foot per track. The total superimposed load exclusive of impact, which may safely be neglected, will then be 6000 lbs. per foot. If this be assumed as distributed over a horizontal distance of 8 ft. it would be equivalent to 750 lbs. per sq. ft. which, if the earth weighs 100 lbs. per cu. ft. would correspond to an additional height of $\frac{750}{100}=7.5$ ft. The pressure on the wall would therefore correspond very closely to that coming from a mass of earth with an upper surface *abcde*.

Rankine's method would not apply exactly in this case since the upper surface is not plane. The error in applying it would, however, probably be on the safe side. The method of trial can be used with no greater error than in the ordinary case. It should be noted that the pressure on the back of a wall in such a case would be represented by a trapezoid rather than a triangle.

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PROBLEMS

74. Determine whether under the worst conditions for overturning the resultant pressure on the base of the conduit shown in Prob. 73 will pass outside the middle third of the base. Assume that earth is filled against outer left side to an elevation of 10 ft. adjoining the conduit, with a surface slope of 1 to 4 upward from the conduit, and that it has the physical properties given in the following problem.

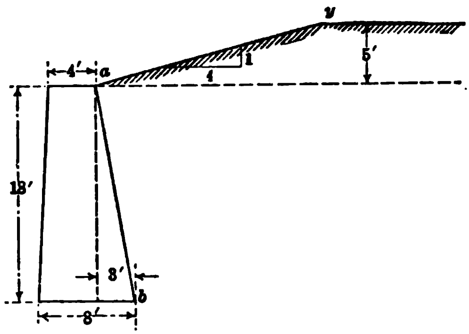
75. Determine magnitude and direction of active earth pressure per ft. of length on side *ab* of this retaining wall by method of trial.

Weight of masonry 150 lbs. per cu. ft.

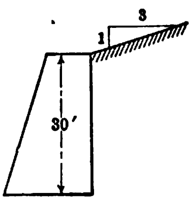
Weight of earth 110 lbs. per cu. ft.

Angle of repose of earth, 30°.

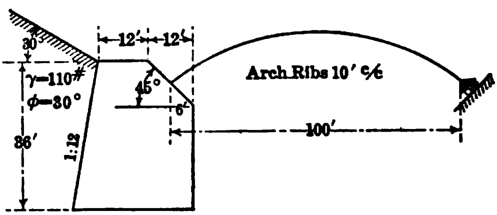
76. The active earth pressure per ft. of length exerted on the vertical back of this retaining wall 30 ft. high is 27,000 lbs. What would the passive pressure be? $\varphi = 30°$.



PROB. 75.



PROB. 76.



PROB. 77.

77. Determine the magnitude and direction (H & V Components) and the point of application of the resultant of the pressure upon that portion of this wall which can be counted upon to resist the thrust of one arch rib.

CHAPTER XX

MASONRY ARCHES WITH FIXED ENDS

220. Description. See Chapter X.

221. Preliminary Determination of Cross-section. To design an arch it is first necessary to make a preliminary determination of its shape and thickness. One method of doing this is as follows:

1st. Assume thickness at crown to correspond with other arches of similar span and loading.

2nd. Determine the curve of the intrados. This should be such as to provide proper clearance over the way or stream, and to give a graceful appearance. A segment of a circle is sometimes employed for flat arches, but it is more customary to use a compound curve, three centred curves being frequently employed. Elliptical arches are also built.

3rd. Assume the thickness of the arch at the skewback. This should be considerably greater than that at crown. Formulas for this thickness are sometimes used, but equally good results may be obtained by the use of the designer's judgment. A thickness of from two to three times that at crown will often give a good preliminary section.

4th. Construct the curve of the extrados so as to make the arch ring symmetrical. This curve should, of course, depend upon the intrados curve.

222. Luten Method of Determining Trial Sections. The following method of determining the shape of the arch ring for earth-covered reinforced concrete arches is given by Daniel B. Luten of the National Bridge Company of Indianapolis, Indiana, and is reproduced here by permission. It is said to give good results.

Let t = crown thickness in inches.
h = rise of arch axis in feet.
L = span in feet.
F = fill over extrados at crown in feet.
P = concentrated load in tons per track applied to half span (for railroad bridge).

U = uniform load in pounds per square foot.
R_i = radius of intrados curve in feet.
R_e = radius of extrados curve in feet.
C = coefficient of friction of concrete on foundation material.
A = area in square feet of abutment below springing line and adjacent to the back tangent.
H = height of intrados at crown above ground in feet.

a. Locate the intrados curve of the arch as follows:

Draw an ellipse of the required span and rise; pass a segment of a circle through crown and springings of ellipse; bisect the

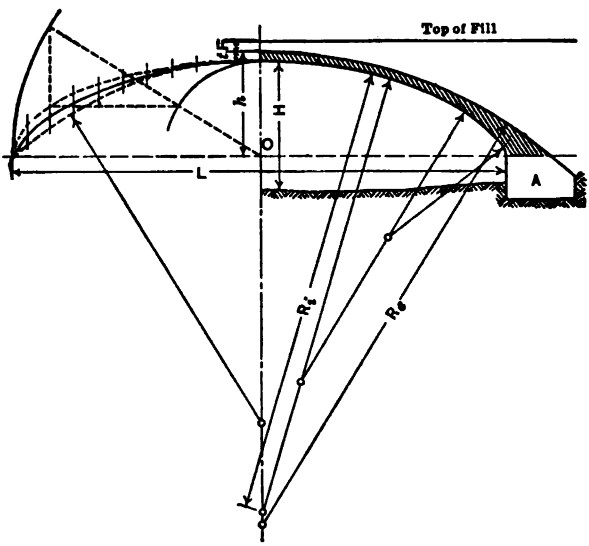


FIG. 344.

vertical distances between the ellipse and the circle; approximate the resulting curve by arcs of circles, adjusting the curve at the springings to become tangent to the verticals. To construct an ellipse, strike two concentric circles with centre at O, Fig. 344, passing respectively through crown and springing and draw radii intersecting these circles. The points of intersection of horizontal and vertical lines drawn through the points of intersection of each radius with the crown circle and springing circle respectively determine points on the ellipse.

b. Apply the formulas which follow to determine the outline of the arch.

$$t=\frac{3L^2(h+3F)}{4000h-L^2}+\frac{UL^2}{30{,}000h}+\frac{P(L+5h)}{150h}+4 \quad \ldots\ldots (77)$$

$$R_e=R_i+\frac{t}{6} \quad \ldots\ldots (78)$$

$$A=\frac{4(2t-H)}{C} \quad \ldots\ldots (79)$$

The extrados should be continued by its tangents to the level of the springing lines.

If uniform live load and concentrated live load are not applied to the arch simultaneously, use only the larger of the terms involving these quantities.

Formula (77) is for a structure intended to be equally strong in all parts; for short spans, however, the constant 4 in equation (77) is said to give a crown thickness which, while desirable from practical considerations, is excessive for the given loading. For short spans, therefore, this term may be neglected in determining t before substituting in formula (79).

223. Outer Forces. After the arch ring is assumed, the loads must be determined. To accomplish this divide the arch ring

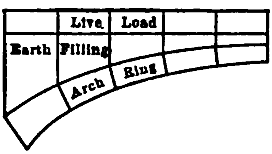


FIG. 345.

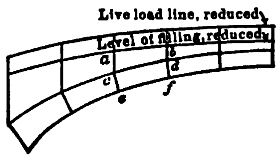


FIG. 346.

into sections of reasonable length, and compute the magnitude and point of application of the load acting on each section, including the weight of the arch itself.

Fig. 345 shows a method of dividing the arch into sections. The dead weight is usually reduced to units corresponding to the weight of the arch ring; e.g., if the filling weighs 120 lbs. per cubic foot and the masonry of the arch ring 150 lbs., a 12 ft. height of filling would be plotted above the arch ring as $12\times\frac{120}{150}=9.6$ ft. The areas and location of the centres of gravity of the figures

abcd and *cefd*, Fig. 346, may next be determined. Both of these areas may be considered as trapezoids. The points of application of the resultant of the weights represented by trapezoids *abcd* and *cdef* must then be determined, either graphically or analytically.[1]

If desired, the live load may be combined with the dead load, both being reduced to an equivalent dead load. Inasmuch as it is necessary to investigate at least two positions of the live load, and sometimes more, it is usually advisable to consider it separately. It is sometimes simpler to determine analytically the position of the point of application of the resultant of the weight of the filling and the arch ring rather than to reduce to common units. Impact is generally disregarded in masonry arches owing to the deadening effect of the filling.

With respect to the position of the live load, it may be said that full live load over the entire arch does not give the worst condition for stability, but may give the maximum fibre stress at some sections. For arches of short span, it is usually considered sufficient to consider full loading and half loading only. For important structures, an influence table or influence lines may be constructed, and the correct position of live load for maximum stresses, minimum stability, etc., determined.

224. Theory.[2] An arch span with fixed ends, i.e., fixed in position and direction, is indeterminate to the third degree with respect to the outer forces. The three unknowns which cannot be determined by statics may either be taken as three of the reaction functions, or as a moment, shear and thrust acting at the crown or at any other convenient section of the arch, or at any convenient point in space. Such unknowns are illustrated by Fig. 348 in which the arch is shown as divided into two parts which are separated for convenience in representation; these unknowns might equally well be considered as acting at any point in space.

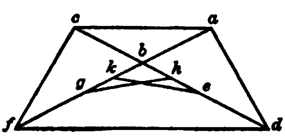


FIG. 347.

[1] To determine the centre of gravity of a trapezoid, the graphical methods already given may be used, but the following method illustrated by Fig. 347 is often more convenient. Lay off $fg = ab$, and $de = cb$. Bisect bg and be. The centre of gravity is at the intersection of ek and gh.

[2] This treatment is based upon the method given by Müller-Breslau in Zeitschrift des Architekten und Ingenieur Vereins, Hannover, 1884.

An exact theory for the arch is complicated and will not be given here. The simple equations which follow are derived by the theorem of least work and are based upon the following assumptions in addition to those commonly made in the development of the beam formula.

1. The distribution of stress over the cross-section of a curved bar is the same as that over the cross-section of a straight bar.

2. The axial thrust for arches having ordinary ratios of rise to span is constant throughout the arch and in the case of symmetrical arches carrying vertical loads is approximately equal to the horizontal component of the crown thrust.[1]

3. Work due to shear is negligible.

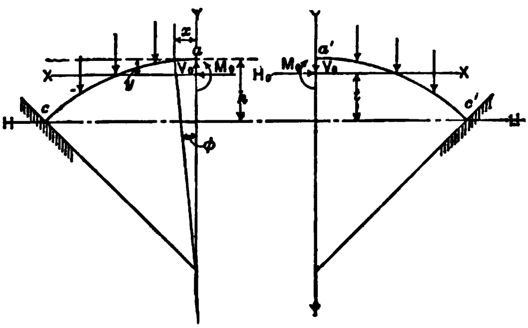


FIG. 348.

With respect to these assumptions it may be stated that the effect of curvature upon the distribution of stress over the cross-section is negligible for curved bars having a ratio of thickness to radius of curvature less than 6/100, i.e. for practically all bridge arches.

The effect of the second assumption is to give an approximate value to the second term in the general equation for work in a bar carrying both direct stress and bending; viz.,

$$W = \int \frac{M^2}{2EI} ds + \int \frac{T^2}{2AE} d \qquad \text{(See article 188)}$$

As the work due to direct thrust in an ordinary arch is small compared with that due to bending and may ordinarily be

[1] The exact expression for axial thrust due to single load of P is as follows: $T = H \cos \varphi + P \sin \varphi$; in which T = axial thrust, H = crown thrust. This evidently is approximately equal to H for small values of φ.

entirely neglected without serious error, an approximation in its value is allowable. Moreover the approximation is fortunately the least over the central portion of the arch where the arch ring is thinnest and the effect of direct thrust greatest.

Making these approximations, the method of least work may now be applied to the symmetrical arch shown diagrammatically in Fig. 348. In all cases moment is to be considered positive when causing compression in upper fibre of arch.

Let XX and YY be two axes of reference, the former being so located that $\int_0^s \frac{yds}{EI} = 0$ and the latter being an axis of symmetry.

x and y be the ordinates in ft. of any point on the arch axis referred to these axes, x being positive to left and y positive upward.

s be the length in ft. of any portion of the arch axis.

M_0 be the moment about an axis normal to paper and passing through the intersection of XX and YY.

H_0 and V_0 be horizontal and vertical forces respectively assumed as acting on each half of arch at intersection of XX and YY.

m_L = numerical value of moment of applied loads to left of any given normal section about any point xy on left half of arch, considering this half to act as a cantilever beam fixed at the abutment.

m_R = similar moment on right half of arch.

M_c = actual bending moment in ft. lbs. about an axis normal to the arch axis at point a.

I = moment of inertia in ft. units of a normal cross-section about an axis normal to the arch axis at point xy.

A = area of a normal cross-section of the arch at point xy.

E = modulus of elasticity in ft. units (E in inch units $\times$ 144).

d = thickness in ft. of normal section at centre of any segment into which arch is divided.

The resulting equations for a fixed ended symmetrical arch of homogeneous material loaded with vertical forces are as follows:

$$M_0=\frac{\int_0^{\frac{s}{2}}(m_L+m_R)\frac{ds}{I}}{2\int_0^{\frac{s}{2}}\frac{ds}{I}} \quad . \quad . \quad . \quad . \quad . \quad (80)$$

$$H_0=-\frac{\int_0^{\frac{s}{2}}(m_L+m_R)\frac{yds}{I}}{2\int_0^{\frac{s}{2}}\frac{y^2ds}{I}+2\int_0^{\frac{s}{2}}\frac{ds}{A}} \quad . \quad . \quad . \quad (81)$$

$$V_0=\frac{\int_0^{\frac{s}{2}}(m_L-m_R)\frac{xds}{I}}{2\int_0^{\frac{s}{2}}x^2\frac{ds}{I}} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Note that m_L and m_R are numerical values for the moment due to downward forces hence for such forces M_O will always be positive. H_O will also always be positive since the axis XX is always nearer the crown than the springing line hence values of Y corresponding to the larger values of m_L and m_R are negative. A negative value for V_o shows that it will act on each half in the opposite direction to that assumed.

Equations 80–82 are referred to the axis XX which may be located with reference to any horizontal axis such as HH, Fig. 348, by application of the auxiliary equation

$$t=\frac{\int z\frac{ds}{I}}{\int\frac{I}{sp}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

in which z is distance from centre of any arc ds to HH.

The equation $Mc=M_0-H_0(h-t)$ gives the moment at the crown.

Equations 80–82 may be derived as follows: In addition to the nomenclature already used

Let M_L = actual moment at any point on left half of arch.

M_R' = corresponding moment on right half of arch.

W = work on entire arch.

Then noting that the moment V_0x is positive when causing compression in upper fibre regardless of sign of x and V_0 and assuming that direct thrust $= H_O$ we get:

$$M_L = M_0 - m_L - H_0 y + V_0 x$$

and $$M_R = M_0 - m_R - H_0 y - V_0 x$$

$$\text{Hence } W = \int_0^{\frac{s}{2}} \frac{M_L^2 ds}{2EI} + \int_{-\frac{s}{2}}^{0} \frac{M_R^2 ds}{2EI} + H_0^2 \int_{-\frac{s}{2}}^{\frac{s}{2}} \frac{ds}{2AE}$$

Substituting values of M_L and M_R previously derived and differentiating with respect to the independent variables gives

$$\frac{dW}{dM_0} = \int_0^{\frac{s}{2}} (M_0 - m_L - H_0 y + V_0 x)\frac{ds}{EI} + \int_{-\frac{s}{2}}^{0} (M_0 - m_R - H_0 y - V_0 x)\frac{ds}{EI}$$

$$\frac{dW}{dH_0} = \int_0^{\frac{s}{2}} (M_0 - m_L - H_0 y + V_0 x)\left(-y\frac{ds}{EI}\right) +$$

$$\int_{-\frac{s}{2}}^{0} (M_0 - m_R - H_0 y - V_0 x)\left(-y\frac{ds}{EI}\right) + H_0 \int_{-\frac{s}{2}}^{\frac{s}{2}} \frac{ds}{AE}$$

$$\frac{dW}{dV_0} = \int_0^{\frac{s}{2}} (M_0 - m_L - H_0 y + V_0 x)\left(\frac{x ds}{EI}\right) +$$

$$\int_{-\frac{s}{2}}^{0} (M_0 - m_R - H_0 y - V_0 x)\left(-x\frac{ds}{EI}\right)$$

Since by construction $\int \frac{y ds}{EI} = 0$, either for the whole and or for half of a symmetrical arch, all terms consisting of the product of this term and a constant may be placed equal to zero. Eliminating such terms therefore and combining other terms noting that $\int_{-\frac{s}{2}}^{0} ds = \int_0^{\frac{s}{2}} ds$, and that $\int_{-\frac{s}{2}}^{\frac{s}{2}} \frac{ds}{AE} = 2\int_0^{\frac{s}{2}} \frac{ds}{AE}$ gives the following expressions:

$$\frac{dW}{dM_0} = 2M_0 \int_0^{\frac{s}{2}} \frac{ds}{EI} - \int_0^{\frac{s}{2}} \frac{(m_L + m_R) ds}{EI} = 0$$

$$\frac{dW}{dH_0} = \int_0^{\frac{s}{2}} \frac{(m_L + m_R) y ds}{EI} + 2H_0 \int_0^{\frac{s}{2}} \frac{y^2 ds}{EI} + 2H_0 \int_0^{\frac{s}{2}} \frac{ds}{AE} = 0$$

$$\frac{dW}{dV_0} = \int_0^{\frac{s}{2}} \frac{(m_R - m_L) x ds}{EI} + 2V_0 \int_0^{\frac{s}{2}} \frac{x^2 ds}{EI} = 0$$

From which equations 80–82 are easily derived.

The auxiliary equation, 83, is obtained by locating the centre of gravity of a set of forces each having the value of $\frac{ds}{I}$. It is evident that the moment of these about any horizontal axis is $\int z\frac{ds}{I}$ in which z is the distance of the centre of each force to the axis. Dividing this by their sum, $\int\frac{ds}{I}$, gives the distance t from the reference axis to an axis passing through their centre of gravity, i.e., one about which

$$\int\frac{yds}{EI}=0$$

As integration of equations 80–83 is difficult it is advisable to divide the arch axis into parts of equal lengths Δs and to replace the integration sign by that of summation. Moreover, in arches of masonry or plain concrete I may be replaced by $\frac{d^3}{12}$ if a slice one ft. in width is considered. This is also sufficiently correct for reinforced concrete arches to make its use possible in a preliminary design. Making these substitutions and cancelling Δs and E from the equations gives the following simple expressions in which the summation sign applies to one-half the arch.

$$M_0=\frac{1}{2}\frac{\Sigma\frac{(m_L+m_R)}{d^3}}{\Sigma\frac{1}{d^3}} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$H_0=-\frac{1}{2}\frac{\Sigma\frac{(m_L+m_R)y}{d^3}}{\Sigma\frac{y^2}{d^3}+\Sigma\frac{1}{12d}} \quad . \quad . \quad . \quad . \quad . \quad (85)$$

$$V_0=\frac{1}{2}\frac{\Sigma\frac{(m_L-m_R)x}{d^3}}{\Sigma\frac{x^2}{d^3}} \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$t=\frac{\Sigma\frac{z}{d^3}}{\Sigma\frac{1}{d^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

225. Formula for Arches of Constant Cross-section. For arches of constant cross-section, d^3 may be cancelled from equa-

tions 84 to 87. If n equals the number of equal parts into which the arch is divided, the equations then become

$$M_0=\frac{1}{2}\,\frac{\Sigma(m_L+m_R)}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

$$H_0=-\frac{1}{2}\,\frac{\Sigma(m_L+m_R)y}{\Sigma y^2+\Sigma\dfrac{d^2}{12}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (89)$$

$$V_0=\frac{1}{2}\,\frac{\Sigma(m_L-m_R)x}{\Sigma x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (90)$$

$$t=\frac{\Sigma z}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (91)$$

In the above equations the summations for x, y, z refer to one-half the arch.

The effect of rib shortening due to direct thrust appears only in the term for H_0 and is due to the last term in the denominator of that expression; the effect of this term upon the value of H_0 is very small since y is always large compared with d.

226. Comparison of Arch and Fixed-ended Beam. Examination of formulas 80 to 82 shows that the expressions for M_0 and V_0 do not contain terms in y and hence are independent of the rise of the arch, from which it follows that the value of these terms should be the same as for a fixed-ended beam having a value of $\frac{ds}{I}$ equal to that of the arch. The accuracy of these equations may therefore be tested by computing M_0 and V_0 for a uniform load applied to a fixed-ended beam of constant cross-section.

For such a beam equations 80 and 82 become for the case a uniform load of w per foot over the left half of the beam

$$M_0=\frac{\displaystyle\int_0^{\frac{L}{2}}\left(\frac{wx^2}{2}\right)dx}{2\displaystyle\int_0^{\frac{L}{2}}dx}=\frac{wL^2}{48}$$

$$V_0=\frac{\displaystyle\int_0^{\frac{L}{2}}\left(\frac{wx^2}{2}\right)xdx}{2\displaystyle\int_0^{\frac{L}{2}}x^2dx}=\frac{3}{32}wL$$

These values are identical with those given by the ordinary formulas for fixed-ended beams of constant section subjected to the same loading.

Had the approximate equations 88 to 91 been used the corresponding values assuming the half arch to be divided into 10 divisions would be as follows:

$$M_0 = \frac{wL^2}{49}$$

$$V_0 = \frac{wL}{10.7}$$

For uniform load over the entire span, the two values of M_0 and V_0 as obtained by formulas 80 and 82 are

$$M_0 = \frac{wL^2}{24}; \qquad V_0 = 0$$

These agree also with the values obtained for the fixed-ended beam.

227. Temperature Stresses. In an arch with fixed ends, the effect of temperature must be considered. The range of temperature to which a masonry arch is subjected is not known precisely, since it depends upon the conductivity of the materials. Numerous investigations upon the subject have been made, and are given in the references at end of chapter. It is generally considered advisable to allow for concrete bridge arches in the latitude of Boston or New York from 20° to 40° range each way from the average temperature.

The change in temperature affects H_0 only, and is given by the following approximate formula in which

ϵ = coefficient of expansion of material.
t_0 = change in temperature in degrees.
n = number of equal parts into which the half arch is divided.

$$H_0 = \frac{\epsilon t_0 n E}{\Sigma\frac{12y^2}{d^3} + \Sigma\frac{1}{d}} \qquad (92)$$

In which the summation refers to one-half of the arch.

In this formula no serious error will occur if the last term in the denominator is omitted.

For an arch of infinite radius, i.e., a beam $y = 0$. $\therefore H_0 =$

$\frac{\epsilon t_o nEd}{n} = \epsilon t_0 Ed$ and stress per sq. ft. $= \frac{H_o}{d} = \epsilon t_0 E$ as should be the case.

For a concrete arch ϵ may be taken as .000006 and E as 288,000,000, hence equation (92) may be written thus for a rib one ft. in width.

$$H_0 = \frac{144nt}{\Sigma\frac{y^2}{d^3} + \frac{1}{12}\Sigma\frac{1}{d}} \tag{93}$$

Equation (92) may be derived by the method of least work as follows:

$$\text{Total work in arch} = \frac{1}{2}\int\frac{M^2ds}{EI} + \frac{1}{2}\int\frac{T^2ds}{AE} - T\epsilon t_o\int ds$$

In which M and T are moment and thrust due to temperature only.

In this expression the last two terms give the work due to direct thrust making due allowance for the fact that the change in length due to temperature is opposite in effect to that due to the thrust caused by the change in temperature.

V_0 due to change in temperature equals zero, hence

$$M_L = M_0 - H_0 y = M_R$$

We may also write

$$T = H_0 \cos\varphi$$

Making these substitutions and differentiating with respect to the two variables, M_0 and H_0, gives the following expression:

$$\frac{dW}{dM_0} = 2\int_0^{\frac{s}{2}}\frac{M_0 ds}{EI} - 2H_0\int_0^{\frac{s}{2}}\frac{yds}{EI}$$

$$\frac{dW}{dH_0} = -2\int_0^{\frac{s}{2}}\frac{(M_0 - H_0 y)yds}{EI} + 2H_0\int_0^{\frac{s}{2}}\frac{\cos^2\varphi ds}{AE} - 2\epsilon t_o\int_0^{\frac{s}{2}}\cos\varphi ds = 0$$

But $ds = \sec\varphi dx$, hence $2\epsilon t_o\int_0^{\frac{s}{2}}\cos\varphi ds = \epsilon t_o L$

Moreover, $\int\frac{yds}{EI} = 0$, hence $M_0 = 0$

$$\therefore H_0 = \pm\frac{1}{2}\frac{\epsilon t_o LE}{\int_0^{\frac{s}{2}}\frac{y^2ds}{I} + \int_0^{\frac{s}{2}}\frac{\cos^2\varphi ds}{A}}$$

H_0 will be a thrust if t_o corresponds to a rise in temperature.

The second term in the denominator is small and for ordinary arches no serious error will be made in substituting for $\cos^2\varphi$ the value of unity. We may also replace I by $\frac{d^3}{12}$ and A by d.

Making these changes, substituting summations over half arch for integrations, and placing $\frac{L}{\Delta s} = 2n$, in which n equals number of parts into which the half arch is divided gives equation (92) in which the denominator is the same as for H_0 due to vertical loads.

228. Effect of Movement of Abutments. The effect of a movement of the abutments upon the horizontal component of the arch thrust may be determined from the well-known equation for the horizontal deflection of a curved bar. This equation, if the assumptions made in Art. 224 are again made, is as follows:

$$\Delta L = -\int_0^L \frac{H dx}{EA} + \int_0^s \frac{M z ds}{EI}. \quad \text{(See Mechanics.)}$$

In which ΔL = change in span due to horizontal component of abutment movement.

H = horizontal component of abutment thrust which is equal by assumption to the normal component of actual thrust at any section.

z = ordinate at any point on the arch axis measured from a horizontal axis corresponding to HH, Fig. 348.

M = bending moment caused by horizontal thrust about an axis normal to paper at same point.

Substituting for M its value Hz gives

$$\Delta L = -H \int_0^L \frac{dx}{AE} + H \int_0^s \frac{z^2 ds}{EI}$$

Substitution of summations for integrations gives the following equation in which the *summation is applied to one-half of arch* in order to obtain terms identical with those developed in previous equations.

$$E \Delta L = -2H \Sigma \frac{\Delta x}{A} + 2H \Sigma \frac{z^2 \Delta s}{I}$$

hence

$$H = \frac{E \Delta L}{2\Sigma \frac{z^2 \Delta s}{I} - 2\Sigma \frac{\Delta x}{A}} \qquad (94)$$

If I and A are replaced by their values in terms of d, the value of H for a rib one ft. in width is given by the following expression.

$$H = \frac{E \Delta L}{2\left[12 \Sigma \frac{z^2 \Delta s}{d^3} - \Sigma \frac{\Delta x}{d}\right]} \qquad (95)$$

For flat arches Δs may be placed $= \Delta x$.

For an arch of infinite radius and constant cross-section; i.e., a beam, $\Delta x = \frac{\Delta s}{2}$ and $z = 0$, hence formula (95) becomes

$$H = -E\Delta L\frac{d}{L}$$

This agrees with the value of H for the axial stress in a straight bar corresponding to a change of ΔL in its length.

229. Effect of a Single Load. Equations 80 to 91 may be applied to a single load and influence lines for M_0, H_0, and V_0 constructed with great ease.

If the unit load is on the left half of the span $m_R = 0$ and $m_L =$ moment of the load, the reverse being true for a load on the right half of the span.

230. Accuracy of Formulas. The formulas developed in this chapter are more accurate than the formulas ordinarily developed in the treatment of arches since they take account from the start of the effect of rib shortening and do not involve the division of the arch so that $\frac{\Delta s}{I}$ is constant with the resulting lack of accuracy near the abutment. They are easy to apply since any draftsman can obtain all necessary data for a symmetrical arch and fill out tables similar to those given in the illustrative example which follows. Moreover, they can be fitted to an unsymmetrical arch with little trouble by properly selecting inclined axes of reference. These axes should be such that

$$\int y \frac{ds}{EI} = 0$$

and that

$$\int \frac{xds}{EI} = 0$$

as is the case for the horizontal and vertical axes used for the symmetrical arch treated.

231. Line of Resistance. It is always advisable to construct a line of resistance for an arch after the values of the various unknowns have been determined. This may be done in a manner similar to that employed in the case of a masonry dam. For a symmetrical arch symmetrically loaded, the resultant crown

thrust to use in constructing the line of resistance equals H_0, is horizontal, and is located at a distance above the axis XX equal to $\frac{M_0}{H_0}$.

For unsymmetrical loading, the point of application of the resultant crown thrust may be obtained from the same formula, but the thrust is no longer horizontal, its slope being given by the expression $\frac{H_0}{V_0}$. Its horizontal component is H_0. With the crown thrust fixed in position and direction, the line of resistance can be readily constructed. It should be kept within the middle third of the arch ring for unreinforced structures, but may be allowed to pass outside the middle third for reinforced concrete, although this is in general undesirable.

232. Distribution of Stress over Cross-section. The maximum fibre stress in an unreinforced arch may be computed by the methods given for masonry dams. Its value depends upon the eccentricity and magnitude of the thrust at a given section, and several sections may have to be tried to determine the limiting condition. If the arch is of reinforced concrete, the maximum stress upon any section, the resultant upon which passes through the middle third, may be found in a similar manner, the steel being assumed as replaced by an equivalent amount of concrete; that is, an area of concrete equal to the product of the steel area and the ratio $\frac{E_s}{E_c}$ which may generally be taken as 15. The resulting section to be dealt with is similar to that shown in Fig. 349, the fins being opposite the steel bars. If the resultant at any section of a reinforced concrete arch passes outside of the middle third, special formulas must be applied for which see works upon reinforced concrete mentioned in bibliography at end of chapter.

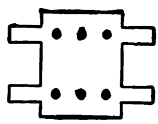


FIG. 349.

233. Computation of External Forces by Approximate Method. *Illustration.* The following example illustrates the method of applying the equations of this chapter to the case of a load over a portion of the span. The arch shown in Fig. 350 is considered and the effect of a uniform load of 200 lbs. per lineal foot over the left half of the span determined. For this loading the values of m_L at the centres of the various sections may be computed analy-

tically by the formula, $m = \frac{wx^2}{2}$, and the value of m_R equals zero for all sections.

Were the problem that of determining the effect of the dead loads the loading should be divided into a series of partial loads as indicated in Art. 223, and the moment at the centre of each section into which the arch is divided computed as if these loads were concentrated loads.

For concentrated live loads, the moment at the centre of each section would be figured in the usual manner for any position of the loads.

The division of the arch into sections of equal length measured along the axis can be readily accomplished as it is unnecessary to

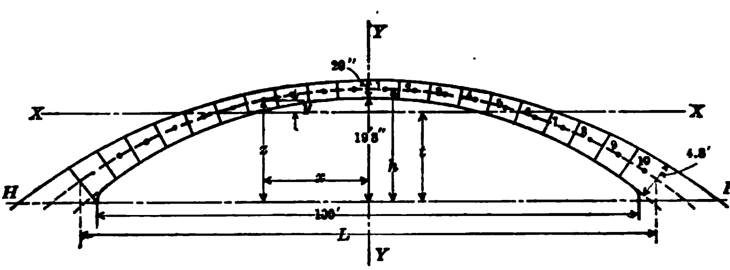


FIG. 350.

work accurately between skewbacks, a variation of a foot or so each way making little difference in the final results. The number of sections into which to divide the arch should be such that no material error will occur from taking the moment throughout each section as constant and equal to that at its centre. The ordinates and thickness of the arch ring given in the tables which follow were scaled from a large sized layout of the arch.

The complete computations are given in the table which follows and require no comment.

With the tabular values once determined, the values of M_0, H_0 and V_0 may be readily obtained from equation 61 to 64, the necessary computations being given at foot of table 5.

TABLE 4

Use for determination of value of t

$$t = \Sigma \frac{z}{d^3} \div \Sigma \frac{1}{d^3}$$

See numerical value at foot of table

Section	z in ft., scaled	d in ft., scaled	d^3	$\frac{1}{d^3}$	$\frac{z}{d^3}$	$\frac{1}{d}$
1	20.49	1.70	4.91	0.204	4.18	0.588
2	20.18	1.70	4.91	0.204	4.12	0.588
3	19.45	1.70	4.91	0.204	3.97	0.588
4	18.45	1.74	5.27	0.190	3.51	0.575
5	17.08	1.75	5.36	0.187	3.19	0.571
6	15.37	1.78	5.64	0.177	2.72	0.562
7	13.38	2.00	8.00	0.125	1.67	0.500
8	10.84	2.45	14.71	0.068	0.74	0.408
9	7.98	3.05	28.37	0.035	0.28	0.328
10	4.70	4.00	64.00	0.016	0.07	0.250
			Total	1.410	24.45	4.958

$$t = \frac{24.45}{1.41} = 17.34 \text{ ft.} \quad h - t = 20.5 - 17.34 = 3.16 \text{ ft.}$$

TABLE 5

For determination of H_0, V_0, and M_0. For equations and values derived therefrom, see foot of table. Moments in 1000-lb. units.

Section	x Scaled	$y = z - t$	x^2	y^2	$\frac{x}{d^3}$	$\frac{y}{d^3}$	$\frac{x^2}{d^3}$	$\frac{y^2}{d^3}$
1	2.84	+ 3.15	8.0	9.9	0.58	+0.642	1.6	2.02
2	8.48	+ 2.84	71.9	8.1	1.73	+0.579	14.7	1.64
3	14.10	+ 2.11	198.8	4.4	2.88	+0.430	40.5	0.91
4	19.68	+ 1.11	387.3	1.2	3.74	+0.211	73.5	0.23
5	25.18	− 0.26	634.0	0.1	4.70	−0.048	118.2	0.01
6	30.55	− 1.97	933.3	3.9	5.41	−0.349	165.5	0.69
7	35.80	− 3.96	1281.6	15.7	4.48	−0.495	160.2	1.96
8	40.89	− 6.50	1672.0	42.2	2.78	−0.442	113.8	2.87
9	45.69	− 9.36	2087.6	87.6	1.61	−0.329	73.6	3.09
10	50.30	−12.64	2530.1	159.8	0.79	−0.198	39.5	2.50
						Total	801.1	15.92

TABLE 5.—(*Continued*)

Live load over left half of arch

Section	m_L	m_R	(m_L+m_R)	(m_L-m_R)	$(m_L+m_R)\frac{1}{d^3}$	$(m_L-m_R)\frac{x}{d^3}$	$(m_L+m_R)\frac{y}{d^3}$
1	0.8	0	0.8	0.8	0.16	0.5	+ 0.5
2	7.2	0	7.2	7.2	1.46	12.4	+ 4.2
3	19.9	0	19.9	19.9	4.05	57.2	+ 8.5
4	38.7	0	38.7	38.7	7.35	144.9	+ 8.2
5	63.4	0	63.4	63.4	11.81	297.8	− 3.0
6	93.3	0	93.3	93.3	16.52	504.0	− 32.6
7	128.2	0	128.2	128.2	16.05	575.0	− 63.5
8	167.2	0	167.2	167.2	11.37	465.0	− 73.9
9	208.8	0	208.8	208.8	7.36	366.0	− 68.6
10	253.0	0	253.0	253.2	3.95	198.5	− 50.2
				Total	80.08	2591.3	−270.4

$$M_0=\frac{1}{2}\,\frac{80{,}100}{1.41}=28{,}400 \text{ ft. lbs.}$$
$$H_0=\frac{1}{2}\,\frac{270{,}400}{15.92+0.41}=8270 \text{ lbs.}$$
$$M_c=34{,}600-8030\times 4.1=+1680 \text{ ft. lbs.}$$
$$V_0=\frac{1}{2}\,\frac{2{,}591{,}300}{801.1}=1615 \text{ lbs.}$$

For temperature stresses $M_0 = 0$ and $n = 10$ hence equation (93) gives $H_0 = \frac{1440}{26.8} = \pm 88$ lbs. per degree F. change in temperature hence $M_c = \pm 88 \times 4.1 = \pm 361$ ft. lbs. per degree F. change in temperature.

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PROBLEMS

78. Compute the value of M_0, H_0 and V_0 for arch shown in Art. 233 for a live load of 200 lbs. per lineal ft. extending over the central half of the arch.

79. Using values of H_0, V_0 and M_0 determined in Art. 233 compute the moment, shear, and normal thrust acting at a section normal to arch axis at a point 30 ft. horizontally from centre of arch, and determine the eccentricity of the thrust at this section.

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